Exceptional surgeries on alternating knots

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joint work with

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K. Ichihara and H. Masai,
Exceptional surgeries on alternating knots,

Related files and outcomes are downloadable at:
http://www.math.chs.nihon-u.ac.jp/~ichihara/ExcAlt/
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3-dimensional manifold (3-manifold)

A space locally modelled on $\mathbb{R}^3$ (like our UNIVERSE)

Curved Spaces by J. Weeks

http://geometrygames.org/CurvedSpaces/index.html
Classification of 3-manifolds

Classification Theorem

Every closed orientable 3-manifold is;

- **Reducible** (containing essential sphere)
- **Toral** (containing essential torus)
- **Seifert fibered** (admitting a foliation by circles)
- **Hyperbolic** (admitting Riem. metric of const. curv. $-1$)

as a consequence of the Geometrization Conjecture including famous Poincaré Conjecture (1904) conjectured by Thurston (late '70s) established by Perelman (2002-03)
• Attack the remaining Open Problems.
  (e.g., “Heegaard genus VS rank of $\pi_1$” problem, . . . )

• Relate Geometric & Topological invariants
  (e.g., Volume conjecture . . . )

• Study the Relationships between 3-mfds.
  (e.g., Dehn surgery . . . )
  (⇑ Today!)
Dehn surgery on a knot

An operation to connect a pair of 3-manifolds

\[ K : \text{a knot in a 3-manifold } M \]

Dehn surgery on \( K \)

1) remove the interior of a regular nbd \( N(K) \) of \( K \) from \( M \) (to obtain the exterior of \( K \))

2) glue a solid torus back (along a slope \( \gamma \))

3-mf; \( M \)  \[ \xrightarrow{\text{Dehn surgery}} \]  \( (K, \gamma) \)  \[ \xrightarrow{\text{Solid torus}} \]
**K** : a knot in the 3-sphere $S^3$

### Notation

For $f : \partial V \to \partial E(K)$ and the meridian $m$ of $V$, the slope (i.e., isotopy class) $\gamma$ of the loop $f(m)$ on $\partial E(K)$ is called the **surgery slope**.

Such a slope on $\partial E(K)$ can be regarded as $r \in \mathbb{Q} \cup \{1/0\}$.

### Notation

$K(r)$: the manifold obtained by surgery on $K$ along $r$. 

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### Motivation

**Hyperbolic Dehn Surgery Theorem [Thurston (1978)]**

Only finitely many Dehn surgeries on a hyperbolic knot (i.e., knot with hyperbolic complement) yield non-hyperbolic manifolds.

**Exceptional surgery**

A Dehn surgery on a hyperbolic knot is called exceptional if it yields a non-hyperbolic manifold.

**Ultimate Goal**

Classify all the exceptional surgeries on hyperbolic knots in the 3-sphere $S^3$. 
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Let $K$ be a hyperbolic alternating knot in $S^3$. If $K$ admits a non-trivial exceptional surgery, then $K$ is equivalent to an arborescent knot.

Facts

Complete classifications of exceptional surgeries have been known for

- 2-bridge knots [Brittenham-Wu, 2001]

Let \( K \) be a hyperbolic alternating knot in \( S^3 \). Suppose that \( K(r) \) is non-hyperbolic for a rational number \( r \). Then \( K(r) \) is irreducible, and the following hold. If \( K(r) \) is toroidal, then \( K(r) \) is not a Seifert fibered, and \( K \) is equivalent to either

- the figure-eight knot and \( r = 0, \pm 4 \),
- a two bridge knot \( K_{[b_1,b_2]} \) with \( |b_1|, |b_2| > 2 \), and \( r = 0 \) if both \( b_1, b_2 \) are even, \( r = 2b_2 \) if \( b_1 \) is odd and \( b_2 \) is even,
- a twist knot \( K_{[2n,\pm 2]} \) with \( |n| > 1 \) and \( r = 0, \mp 4 \),
- a pretzel knot \( P(q_1,q_2,q_3) \) with \( q_i \neq 0, \pm 1 \) for \( i = 1, 2, 3 \), and \( r = 0 \) if \( q_1, q_2, q_3 \) are all odd, \( r = 2(q_2 + q_3) \) if \( q_1 \) is even and \( q_2, q_3 \) are odd.

In the above, when \( r \neq 0 \), then \( r \) is always a boundary slope of a once punctured Klein bottle spanned by \( K \).

If $K(r)$ is small Seifert fibered, then $K(r)$ has the infinite fundamental group, and $K$ is equivalent to either

- the figure-eight knot and $r = \pm 1, \pm 2, \pm 3$,
- a twist knot $K[2n,\pm 2]$ with $|n| > 1$ and $r = \mp 1, \pm 2, \pm 3$.

In particular, the figure-eight knot is the only knot admitting 10 exceptional surgeries among hyperbolic alternating knots, and the others admit at most 5 exceptional surgeries.
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Let $K$ be a hyperbolic alternating knot in $S^3$.

**Fact [Lackenby (2000)]**

If $K$ has a prime alternating diagram $D$ satisfying $t(D) \geq 9$, then $K$ admits no exceptional surgeries.

Here $t(D)$ denotes the number of twists.

A twist is defined as either;

- a maximal connected collection of bigon regions in $D$
- or an isolated crossing adjacent to no bigon regions.
• All alternating knots with $t(D) \leq 5$ are arborescent.
• Most alternating knots with $t(D) \leq 8$ are arborescent.

A knot $K$ is called an **arborescent** knot if it can be obtained by summing and gluing several rational tangles together.

**Fact (Brittenham-Wu, Wu, I.-Jong, Meier)**

All the exceptional surgeries on hyperbolic alternating arborescent knots are completely classified.
Lemma

Suppose that a hyperbolic alternating knot $K$ in $S^3$ has a connected prime alternating diagram $D$ satisfying $t(D) \leq 8$.

Moreover suppose that $K$ is not an arborescent knot.

Then the diagram $D$ satisfies $6 \leq t(D) \leq 8$
and is obtained from one of the 9 plane graphs
by substituting one of the 4 tangles in the next Figure
to each of the fat vertices in the graphs,
and performing twistings on all the augmented circles.
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Thus, for example, $K$ can be obtained by twisting along augmented loops from the following links...
Thus, for example, $K$ can be obtained by twisting along augmented loops from the following links...
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Thus, for example, $K$ can be obtained by twisting along augmented loops from the following links...
It suffice to get a complete classification of exceptional surgeries on the links like those as illustrated.

The number of such links are at most

\[4^6 + 4^7 + 7 \times 4^8 = 479232\]

By using symmetry, and other restrictions, we reduce the number into 30404.

We listed up the links by using computer.

Target: Exceptional surgeries on these 30404 links.
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To study exceptional surgeries on the links, we further used a computer program developed in:

B. Martelli, C. Petronio, F. Roukema

**Exceptional Dehn surgery on the minimally twisted five-chain link**

preprint, arXiv:1109.0903

The program relies upon

- **SnapPy** (based on **SnapPea**): computer software calculates various hyperbolic invariants for 3-manifolds.
We modified the original codes to use interval arithmetics (explained in the previous talk) and applied the program \texttt{hikmot} developed in

N. Hoffman, K. Ichihara, M. Kashiwagi, H. Masai, S. Oishi, and A. Takayasu,
http://www.oishi.info.waseda.ac.jp/~takayasu/hikmot/

It can possibly give us a rigorous complete classification of exceptional surgeries on a given hyperbolic link.
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Computation time

- We have **30404** links to investigate.
- For ONE link, in the worst case, we need around **9 HOURs** by PC.

⇒ We need to reduce the computation time.

**Fact [Wu] + Observation**

Suppose
- the edges go through a crossing circle are anti-parallel
- after "smoothing", #(knot components) > 1.

Then the links obtained by twisting more than once along augmented loops have NO exceptional surgeries.

⇒ We can reduce the totally computation time to about 1/100.

Still we need more than **2000** hours by PC... (about 81 days)
• TSUBAME is the supercomputer of Tokyo Tech.
• Intuitively we can use many machines simultaneously.
• I ”rent” 320 machines.
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Main procedure

feito: (based on find_exceptional_filling.py)

Input: A triangulation data $T$ of a manifold $N$
Output: A verification that all non-trivial Dehn surgeries of $N$ satisfying some conditions are hyperbolic.

Given a triangulation of a 3-manifold, find a hyperbolic structure via hikmot, list the slopes of length $< 6.0001$ up, and perform surgeries along the slopes. Do this procedure recursively.

Recall that:

The 6-theorem [Agol, Lackenby (2000)]
A Dehn surgery along a slope of length $> 6$ is not exceptional.
Input: an ideal **triangulation** of a cusped 3-manifold

(1) 

Try to find a nicely approximated hyperbolic structure, and try to certify it via hikmot.

Can such a solution be found? Does the obtained triangulation contain flat or nearly flat tetrahedra?

[ Y. → (3) ] [ N. → (2) ]
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**Procedure**

(2) Randomize the triangulation.

[ \rightarrow (1) ]

(3) Find and fix a nearly maximal cusp by horoball expansions.
List up the slopes of length \(< 6.0001\) on the cusp.
Do such slopes exist?

[ Y. \rightarrow (4) ] [ N. \rightarrow (end) ]
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**Procedure**

(4)

Perform a Dehn filling along a slope in the list.

Find a nicely approximated hyperbolic structure, and certify it via hikmot.

Can such a solution be found?

\[
\text{Y. } \rightarrow \text{(6)} \quad \text{[ N. } \rightarrow \text{(5)} \text{]}
\]

(5)

Randomize the triangulation.

\[
\rightarrow \text{(4)}
\]
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(6)

Recursively apply this procedure for the obtained cusped hyperbolic 3-manifolds.

We have applied our main code \texttt{fef.py}.

Actually, applying this procedure, we treated 5868836 hyperbolic 3-manifolds...

Finally we can conclude there are no exceptional surgeries on hyperbolic non-arborescent alternating knots with a prime alternating diagram \( D \) satisfying \( t(D) \leq 8 \).
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The Montesinos knots $M(-1/2, 2/5, 1/(2q + 1))$ with $q \geq 5$ have no non-trivial exceptional surgeries.

Together with known results, this gives the final step in a complete classification of exceptional surgery on arborescent knots as follows.

Let $K$ be a hyperbolic arborescent knot in $S^3$. Suppose that $K(r)$ is non-hyperbolic for $r \in \mathbb{Q}$.

Then $r$ must be an integer except for $r = 37/2$ for $P(-2, 3, 7)$.

The manifold $K(r)$ is always irreducible, and

$\pi_1(K(r))$ is infinite except for

$r = 17, 18, 19$ for $P(-2, 3, 7)$ and $r = 22, 23$ for $P(-2, 3, 9)$. 

If \( K(r) \) is toroidal, then \( K(r) \) is not a Seifert fibered, and \( K \) is equivalent to

- a two bridge knot \( K_{[b_1,b_2]} \) with \( |b_1|, |b_2| > 2 \), and \( r = 0 \) if both \( b_1, b_2 \) are even, \( r = 2b_2 \) if \( b_1 \) is odd and \( b_2 \) is even,
- a twist knot \( K_{[2n,\pm2]} \) with \( |n| > 1 \) and \( r = 0, \mp 4 \),
- one of the Montesinos knots of length 3 with the slope described in Table 1.

- \( K_1 \) with \( r = 3 \), \( K_2 \) with \( r = 0 \) or \( K_3 \) with \( r = -3 \) for
  \[
  (S^3, K_1) = T(1/3, -1/2; 4) \cup \eta T(1/3, -1/2; 4),
  (S^3, K_2) = T(1/3, -1/2; 4) \cup \eta T(-1/3, 1/2; -4), \text{ and}
  (S^3, K_3) = T(-1/3, 1/2; -4) \cup \eta T(-1/3, 1/2; -4).\]
Table: Toroidal surgeries

<table>
<thead>
<tr>
<th>$K$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(q_1, q_2, q_3), q_i \text{ odd and }</td>
<td>q_i</td>
</tr>
<tr>
<td>$P(q_1, q_2, q_3), q_1 \text{ even, } q_2, q_3 \text{ odd and }</td>
<td>q_i</td>
</tr>
<tr>
<td>$P(-2, 3, 7)$</td>
<td>37/2</td>
</tr>
<tr>
<td>$P(-3, 3, 7)$</td>
<td>1</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(3 + 1/n)), n \text{ even and } n \neq 0$</td>
<td>$2 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(5 + 1/n)), n \text{ even and } n \neq 0$</td>
<td>$1 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(6 + 1/n)), n \neq 0, -1 \text{ odd (resp. even)}$</td>
<td>16 (resp. 0)</td>
</tr>
<tr>
<td>$M(-1/2, 1/5, 1/(3 + 1/n)), n \text{ even and } n \neq 0$</td>
<td>$5 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 2/5, 1/7)$</td>
<td>12</td>
</tr>
<tr>
<td>$M(-1/2, 2/5, 1/9)$</td>
<td>15</td>
</tr>
<tr>
<td>$M(-1/3, -1/(3 + 1/n), 2/3), n \neq 0, -1 \text{ odd (resp. even)}$</td>
<td>$-12$ (resp. 4)</td>
</tr>
<tr>
<td>$M(-2/3, 1/3, 1/4)$</td>
<td>13</td>
</tr>
<tr>
<td>$M(-1/(2 + 1/n), 1/3, 1/3), n \text{ odd and } n \neq -1$</td>
<td>$2n$</td>
</tr>
</tbody>
</table>

If $K(r)$ is small Seifert fibered, then $K$ is either

- the figure-eight knot and $r = \pm 1, \pm 2, \pm 3$,
- a twist knot $K_{[2n,\pm]}$ with $|n| > 1$ and $r = \mp 1, \mp 2, \mp 3$,
- one of the Montesinos knots of length 3 with the slope described in Table 2.
### Table: Seifert fibered surgeries

<table>
<thead>
<tr>
<th>$K$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(-2, 3, 2n + 1), n \neq 0, 1, 2$</td>
<td>$4n + 6, 4n + 7$</td>
</tr>
<tr>
<td>$P(-2, 3, 7)$</td>
<td>17</td>
</tr>
<tr>
<td>$P(-3, 3, 3)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 4)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 5)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 6)$</td>
<td>1</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/5)$</td>
<td>$3, 4, 5$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/7)$</td>
<td>$-1, 0, 1$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/9)$</td>
<td>$2, 3, 4$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/11)$</td>
<td>$-1, -2$</td>
</tr>
<tr>
<td>$M(-1/2, 1/5, 2/5)$</td>
<td>7, 8</td>
</tr>
<tr>
<td>$M(-1/2, 1/7, 2/5)$</td>
<td>11</td>
</tr>
<tr>
<td>$M(-2/3, 1/3, 2/5)$</td>
<td>-5</td>
</tr>
</tbody>
</table>
Thank you for your attention!

ありがとうございました。

Grazie mille!