Exceptional surgeries on alternating knots

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joint work with

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K. Ichihara and H. Masai,
Exceptional surgeries on alternating knots,

Related files and outcomes are downloadable at:
http://www.math.chs.nihon-u.ac.jp/~ichihara/ExcAlt/
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3-manifold is ...

3-dimensional manifold (3-manifold)
A space locally modelled on $\mathbb{R}^3$ (like our UNIVERSE)

Curved Spaces by J. Weeks
http://geometrygames.org/CurvedSpaces/index.html
Classification of 3-manifolds

FACT

Every closed orientable 3-manifold is;

- Reducible (containing essential sphere)
- Toroidal (containing essential torus)
- Seifert fibered (admitting a foliation by circles)
- Hyperbolic (admitting Riemetric of const.curv.$-1$)

as a consequence of the Geometrization Conjecture

including famous Poincaré Conjecture (1904)

conjectured by Thurston (late '70s)

established by Perelman (2002-03)
What’s the NEXT?

► Attack the remaining Open Problems.
  (e.g., “Heegaard genus VS rank of $\pi_1$” problem, …)

► Relate Geometric & Topological invariants
  (e.g., Volume conjecture …)

► Study the Relationships between 3-manifolds.
  (e.g., Dehn surgery …)
  (⇑ Today!)
Dehn surgery on a knot

An operation to connect a pair of 3-manifolds

\( K \): a knot (i.e., embedded circle) in a 3-manifold \( M \)

**Dehn surgery on** \( K \)

1) remove the open neighborhood of \( K \) from \( M \) 
   (to obtain the exterior \( E(K) \) of \( K \))

2) glue a solid torus back (along a slope \( \gamma \))
Motivation

Hyperbolic Dehn Surgery Theorem [Thurston (1978)]

Only **finitely many** Dehn surgeries on a **hyperbolic** knot (i.e., knot with hyperbolic complement) yield **non-hyperbolic** manifolds.

Exceptional surgery

A Dehn surgery on a **hyperbolic** knot is called **exceptional** if it yields a **non-hyperbolic** manifold.

Ultimate Goal

Classify all the exceptional surgeries on hyperbolic knots in the 3-sphere $S^3$.

Remark: \[ S^3 = \mathbb{R}^3 \cup \{\infty\} \]
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Result


Let $K$ be a hyperbolic alternating knot in $S^3$. If $K$ admits a non-trivial exceptional surgery, then $K$ is equivalent to an arborescent knot.

Facts

Complete classifications of exceptional surgeries have been known for

- 2-bridge knots [Brittenham-Wu, 2001]

Let $K$ be a hyperbolic alternating knot in $S^3$. Suppose that $K(r)$ is non-hyperbolic for a rational number $r$. Then $K(r)$ is irreducible, and the following hold. If $K(r)$ is toroidal, then $K(r)$ is not a Seifert fibered, and $K$ is equivalent to either

- the figure-eight knot and $r = 0, \pm 4$,
- a two bridge knot $K_{[b_1,b_2]}$ with $|b_1|, |b_2| > 2$, and $r = 0$ if both $b_1, b_2$ are even, $r = 2b_2$ if $b_1$ is odd and $b_2$ is even,
- a twist knot $K_{[2n,\pm 2]}$ with $|n| > 1$ and $r = 0, \mp 4$,
- a pretzel knot $P(q_1, q_2, q_3)$ with $q_i \neq 0, \pm 1$ for $i = 1, 2, 3$, and $r = 0$ if $q_1, q_2, q_3$ are all odd, $r = 2(q_2 + q_3)$ if $q_1$ is even and $q_2, q_3$ are odd.

In the above, when $r \neq 0$, then $r$ is always a boundary slope of a once punctured Klein bottle spanned by $K$. 

If $K(r)$ is small Seifert fibered, then $K(r)$ has the infinite fundamental group, and $K$ is equivalent to either

- the figure-eight knot and $r = \pm 1, \pm 2, \pm 3$,

- a twist knot $K[2n, \pm 2]$ with $|n| > 1$ and $r = \mp 1, \mp 2, \mp 3$.

In particular, the figure-eight knot is the only knot admitting 10 exceptional surgeries among hyperbolic alternating knots, and the others admit at most 5 exceptional surgeries.
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Upper Bound of Twist Number

Let $K$ be a hyperbolic alternating knot in $S^3$.

Fact [Lackenby (2000)]

If $K$ has a prime alternating diagram $D$ satisfying $t(D) \geq 9$, then $K$ admits no exceptional surgeries.

Here $t(D)$ denotes the number of twists.

A twist is defined as either:
- a maximal connected collection of bigon regions in $D$
- or an isolated crossing adjacent to no bigon regions.
A knot $K$ is called an arborescent knot if it can be obtained by summing and gluing several rational tangles together.

**Fact (Brittenham-Wu, Wu, I.-Jong, Meier)**

All the exceptional surgeries on hyperbolic alternating arborescent knots are completely classified.
Lemma

Suppose that a hyperbolic alternating knot $K$ in $S^3$ has a connected prime alternating diagram $D$ satisfying $t(D) \leq 8$. Moreover suppose that $K$ is not an arborescent knot. Then the diagram $D$ satisfies $6 \leq t(D) \leq 8$ and ...
Remaining cases

and is obtained from one of the 9 plane graphs

by substituting one of the 4 tangles

and performing twistings on all the augmented circles.
Thus, for example, $K$ can be obtained by twisting along augmented loops from the following links...
Thus, for example, $K$ can be obtained by twisting along augmented loops from the following links...
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**Remaining cases**

It suffice to get a complete classification of exceptional surgeries on the links like those as illustrated.

The number of such links are at most

\[ 4^6 + 4^7 + 7 \times 4^8 = 479232 \]

By using symmetry, and other restrictions, we reduce the number into 30404.

We listed up the links by using computer.

Target: Exceptional surgeries on these 30404 links.
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Base code

To study exceptional surgeries on the links, we further used a computer program developed in:

B. Martelli, C. Petronio, F. Roukema

**Exceptional Dehn surgery on the minimally twisted five-chain link**
preprint, arXiv:1109.0903

The program relies upon

- **SnapPy** (based on **SnapPea**): computer software calculates various hyperbolic invariants for 3-manifolds.
  
  [http://www.math.uic.edu/t3m/SnapPy/](http://www.math.uic.edu/t3m/SnapPy/)
Ingredients

We modified the original codes to use interval arithmetics (explained in the previous talk) and applied the program hikmot developed in

Hoffman, Ichihara, Kashiwagi, Masai, Oishi, and Takayasu


http://www.oishi.info.waseda.ac.jp/~takayasu/hikmot/

It can possibly give us a rigorous complete classification of exceptional surgeries on a given hyperbolic link.
We have **30404** links to investigate.

Our code applies *hikmot* recursively.

In the worst case, for a single link, there are about **18,000** manifolds to investigate.

⇒ It takes around **51 HOURs** by single CPU.

**We need a high spec machine!!**
TSUBAME

- TSUBAME is the supercomputer of Tokyo Tech. providing large-scale parallel computing.

- In general, to use parallel computation effectively, we need some more work, but in our case, the situation itself is totally parallel, i.e., we need to investigate each link independently.

In total, i.e. the sum of the computation time of all nodes, computation time was approximately \( 512 \) days, and the number of manifolds we applied hikmot is \( 5,646,646 \).
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The Montesinos knots $M(-1/2, 2/5, 1/(2q + 1))$ with $q \geq 5$ have no non-trivial exceptional surgeries.

Together with known results, this gives the final step in a complete classification of exceptional surgery on arborescent knots as follows.

Let $K$ be a hyperbolic arborescent knot in $S^3$. Suppose that $K(r)$ is non-hyperbolic for $r \in \mathbb{Q}$.

Then $r$ must be an integer except for $r = 37/2$ for $P(-2, 3, 7)$.

The manifold $K(r)$ is always irreducible, and $\pi_1(K(r))$ is infinite except for $r = 17, 18, 19$ for $P(-2, 3, 7)$ and $r = 22, 23$ for $P(-2, 3, 9)$. 
Appendix


If $K(r)$ is toroidal, then $K(r)$ is not a Seifert fibered, and $K$ is equivalent to

- a two bridge knot $K_{[b_1, b_2]}$ with $|b_1|, |b_2| > 2$, and $r = 0$ if both $b_1, b_2$ are even, $r = 2b_2$ if $b_1$ is odd and $b_2$ is even,

- a twist knot $K_{[2n, \pm 2]}$ with $|n| > 1$ and $r = 0, \mp 4$,

- one of the Montesinos knots of length 3 with the slope described in Table 1.

- $K_1$ with $r = 3$, $K_2$ with $r = 0$ or $K_3$ with $r = -3$ for

$$(S^3, K_1) = T(1/3, -1/2; 4) \cup T(-1/3, 1/2; -4),$$

$$(S^3, K_2) = T(1/3, -1/2; 4) \cup T(-1/3, 1/2; -4),$$

and

$$(S^3, K_3) = T(-1/3, 1/2; -4) \cup T(-1/3, 1/2; -4).$$
Appendix

Table: Toroidal surgeries

<table>
<thead>
<tr>
<th>$K$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(q_1, q_2, q_3), q_i$ odd and $</td>
<td>q_i</td>
</tr>
<tr>
<td>$P(q_1, q_2, q_3), q_1$ even, $q_2, q_3$ odd and $</td>
<td>q_i</td>
</tr>
<tr>
<td>$P(-2, 3, 7)$</td>
<td>$37/2$</td>
</tr>
<tr>
<td>$P(-3, 3, 7)$</td>
<td>1</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(3 + 1/n)), n$ even and $n \neq 0$</td>
<td>$2 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(5 + 1/n)), n$ even and $n \neq 0$</td>
<td>$1 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 1/(6 + 1/n)), n \neq 0, -1$ odd (resp. even)</td>
<td>16 (resp. 0)</td>
</tr>
<tr>
<td>$M(-1/2, 1/5, 1/(3 + 1/n)), n$ even and $n \neq 0$</td>
<td>$5 - 2n$</td>
</tr>
<tr>
<td>$M(-1/2, 2/5, 1/7)$</td>
<td>12</td>
</tr>
<tr>
<td>$M(-1/2, 2/5, 1/9)$</td>
<td>15</td>
</tr>
<tr>
<td>$M(-1/3, -1/(3 + 1/n), 2/3), n \neq 0, -1$ odd (resp. even)</td>
<td>$-12$ (resp. 4)</td>
</tr>
<tr>
<td>$M(-2/3, 1/3, 1/4)$</td>
<td>13</td>
</tr>
<tr>
<td>$M(-1/(2 + 1/n), 1/3, 1/3), n$ odd and $n \neq -1$</td>
<td>$2n$</td>
</tr>
</tbody>
</table>
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If \( K(r) \) is small Seifert fibered, then \( K \) is either
- the figure-eight knot and \( r = \pm 1, \pm 2, \pm 3 \),
- a twist knot \( K[2n, \pm 2] \) with \( |n| > 1 \) and \( r = \mp 1, \mp 2, \mp 3 \),
- one of the Montesinos knots of length 3 with the slope described in Table 2.
### Appendix

**Table: Seifert fibered surgeries**

<table>
<thead>
<tr>
<th>$K$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(-2, 3, 2n + 1), n \neq 0, 1, 2$</td>
<td>$4n + 6, 4n + 7$</td>
</tr>
<tr>
<td>$P(-2, 3, 7)$</td>
<td>17</td>
</tr>
<tr>
<td>$P(-3, 3, 3)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 4)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 5)$</td>
<td>1</td>
</tr>
<tr>
<td>$P(-3, 3, 6)$</td>
<td>1</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/5)$</td>
<td>$3 , 4 , 5$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/7)$</td>
<td>$-1 , 0 , 1$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/9)$</td>
<td>$2 , 3 , 4$</td>
</tr>
<tr>
<td>$M(-1/2, 1/3, 2/11)$</td>
<td>$-1 , -2$</td>
</tr>
<tr>
<td>$M(-1/2, 1/5, 2/5)$</td>
<td>$7 , 8$</td>
</tr>
<tr>
<td>$M(-1/2, 1/7, 2/5)$</td>
<td>11</td>
</tr>
<tr>
<td>$M(-2/3, 1/3, 2/5)$</td>
<td>$-5$</td>
</tr>
</tbody>
</table>
Thank you for your attention!

Danke schön!