Non left-orderable surgeries and generalized Baumslag-Solitar relators

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**$L$-space Conjecture**

$L$-space Conjecture [Boyer-Gordon-Watson, 2011]

$M$: an irreducible rational homology sphere

$M$ is an $L$-space if and only if $\pi_1(M)$ is not LO

**left-orderability**

A non-trivial group $G$ is called **left-orderable (LO)** if $\exists <$: a strict total order on $G$ which is left invariant:

$g < h \quad \rightarrow \quad fg < fh \quad \text{for } \forall f, g, h \in G$

**$L$-space**

A rational homology sphere $M$ is called an $L$-space if $\text{rk} \hat{HF}(M) = |H_1(M; \mathbb{Z})|$ holds for $\hat{HF}(M)$: Heegaard Floer homology.
**Dehn surgery**

Dehn surgery is one of the simple ways to construct \( L \)-spaces.

The following operation to obtain another 3-manifold from a given 3-manifold is called a **Dehn surgery**.

\( K \): a knot in a 3-manifold \( M \)

**Dehn surgery on** \( K \)

1. remove an open regular neighborhood of \( K \) from \( M \) (drilling)
2. glue a solid torus \( V \) back along a slope \( \frac{p}{q} \) (Dehn filling)
**Left-orderable surgery and $L$-space surgery**

$K$: a knot in 3-sphere $S^3$

$K(p/q)$: a 3-manifold obtained by Dehn surgery on $K$ along the slope $p/q$

**left-orderable surgery**

A Dehn surgery on $K$ is called a **non left-orderable surgery** if it yields a closed 3-manifold with $\pi_1(K(p/q))$ is non left-orderable.

**$L$-space surgery**

A Dehn surgery on $K$ is called an **$L$-space surgery** if it yields a closed 3-manifold which is an $L$-space.

**Question**

Which knots in $S^3$ have non-LO and/or $L$-space surgery?
Known results - Pretzel knots -

**Theorem [Lidman-Moore, preprint (arXiv:1306.6707v1)]**

For $s \geq 3$, only $(-2, 3, 2s + 1)$-pretzel knots have $L$-space surgeries among hyperbolic pretzel knots.

Hence, if $L$-space Conjecture is true, among hyperbolic pretzel knots, only $(-2, 3, 2s + 1)$-pretzel knots would have non-LO surgeries.
Known results - Pretzel knots -

Theorem [Nakae, Clay-Watson, 2013]
For $s \geq 3$, $(-2, 3, 2s + 1)$-pretzel knots have non left-orderable surgeries.

Corollary
If a $(-2, 3, 2s + 1)$-pretzel knot has an $L$-space surgery, then it has a non left-orderable surgery.

Remark: It is still open whether the opposite statement holds.
As an extension of Nakae’s result, we have:

**Theorem [Ichihara-Temma, 2014]**

Let $K$ be a knot in a 3-manifold $M$. Suppose that $\pi_1(M - K)$ has a presentation such as

$$\langle a, b \mid (w_1a^mw_1^{-1})b^{-r}(w_2^{-1}a^nw_2)b^{r-k}\rangle$$

with $m, n \geq 0$, $r \in \mathbb{Z}$, $k \geq 0$, and $a$: a meridian of $K$. Suppose that the longitude of $K$ is represented as $a^{-s}wa^{-t}$

with $s, t \in \mathbb{Z}$ and $w$ is a word without $a^{-1}, b^{-1}$.

If $q \neq 0$ and $p/q \geq s + t$, then Dehn surgery on $K$ along the slope $p/q$ yields a closed 3-manifold with $\pi_1(K(p/q))$ is non left-orderable.
Remark:
The relator in the presentation in Theorem can be regarded as a generalization of the well-known Baumslag-Solitar relator.

The Baumslag-Solitar relator is the relator \( x^{-n} y x^m y^{-1} \) with \( m, n \neq 0 \) in the group generated by \( x, y \).

It plays an important role and is well-studied in combinatorial group theory and geometric group theory. For example;

Theorem [Shalen, 2001]
The Baumslag-Solitar relator cannot appear in the fundamental group of an orientable 3-manifold.
Known results - Twisted Torus knots -

Note:

\((-2, 3, 2s + 1)\)-pretzel knots = twisted torus knots \(K(3, 5; 2, s - 2)\).

Twisted torus knot \(K(3, -4; 2, 2)\)
**Known results - Twisted Torus knots -**

**Theorem [Vafaee, 2014]**

For $p \geq 2$, $k \geq 1$, $r > 0$ and $0 < s < p$,

$K(p, kp \pm 1; s, r)$ has an $L$-space surgery

if and only if either $s = p - 1$ or $s \in \{2, p - 2\}$ and $r = 1$.

**Corollary**

$K(3, q; 2, s)$ has an $L$-space surgery if $q > 0$ and $s \geq 1$.

**Theorem [Clay-Watson, 2013]**

$K(3, 3k + 2; 2, s)$ has a non left-orderable surgery

if (1) $k \geq 0$ and $s = 1$, or (2) $k = 1$ and $s \geq 0$. 
Corollary [Ichihara-Temma, 2014]

For $k, s \geq 0$, $K(3, 3k + 2; 2, s)$ has a non left-orderable surgery.

Precisely $\pi_1(K(p/q))$ is non left-orderable if $p/q \geq 3(3k + 2) + 2s$. 
Recent extensions

Our results have been extended as follows.

**Theorem (Christianson-Goluboff-Hamann-Varadaraj)**

For $p, k, s > 0$, $K(p, pk \pm 1; p - 1, s)$ and $K(p, pk \pm 1; p - 2, 1)$ have non left-orderable surgeries.

This is obtained in Columbia University math REU program by undergraduates.

**Corollary**

For $s > 0$, $K(3, q; 2, s)$ have non left-orderable surgeries.

**Corollary**

If $K(3, q; 2, s)$ has an $L$-space surgery, then it has a non left-orderable surgery.
Left-orderability

Set \( G := \pi_1(K(p/q)) \).

The following is well-known for experts:

**Theorem**

A countable group \( G \) is left-orderable if and only if \( G \) is isomorphic with a subgroup of \( Homeo^+(\mathbb{R}) \).

It suffice to study a homomorphism \( \varphi : G \to Homeo^+(\mathbb{R}) \).
Sample calculations

Abusing notations, we will denote the image of \( \varphi(g)(x) = gx \) for \( g \in G \).

\[
w_1 a^m w_1^{-1} b^{-r} w_2^{-1} a^n w_2 b^{r-k} = 1
\]

\[
\Rightarrow a^n w_2 b^{r-k} w_1 a^m = w_2 b^r w_1
\]

Assume: \( x < ax \) for any \( x \in \mathbb{R} \)

\[
a^n w_2 b^{r-k} w_1 a^m x = w_2 b^r w_1 x
\]
\[
< w_2 b^r w_1 a^m x
\]
\[
< a^n w_2 b^r w_1 a^m x
\]

\[
b^{r-k} x < b^r x \Rightarrow x < b^k x \Rightarrow x < bx \quad (\forall x \in \mathbb{R})
\]