Z-coloring with five colors for links

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Let $L$ be a link, and $D$ a diagram of $L$.

**$\mathbb{Z}$-coloring**

A map $\gamma : \{\text{arcs of } D\} \to \mathbb{Z}$ is called a **$\mathbb{Z}$-coloring** on $D$ if it satisfies the condition $2\gamma(a) = \gamma(b) + \gamma(c)$ at each crossing of $D$ with the over arc $a$ and the under arcs $b$ and $c$.

A $\mathbb{Z}$-coloring which assigns the same color to all the arcs of the diagram is called the **trivial $\mathbb{Z}$-coloring**.
Example
**Z-colorable link**

$L$ is **Z-colorable** if $\exists$ a diagram of $L$ with a non-trivial $\mathbb{Z}$-coloring.

**Remark**

$L$ is **Z-colorable** $\iff \det(L) = 0$

Any knot $K$ is non-$\mathbb{Z}$-colorable since $\det(K)$ is odd.
Let $L$ be a $\mathbb{Z}$-colorable link. Let us consider the cardinality of the image of a non-trivial $\mathbb{Z}$-coloring on a diagram of $L$.

**Minimal coloring number**

We call the minimum of such cardinalities among all non-trivial $\mathbb{Z}$-colorings on diagrams of $L$ the *minimal coloring number* of $L$, and denote it by $\text{mincol}_\mathbb{Z}(L)$. 
Let $L$ be a $\mathbb{Z}$-colorable link.

**Theorem 1**

If $L$ is non-splittable, then $\text{mincol}_\mathbb{Z}(L) \geq 4$. 

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**Proposition**

If the crossing number of $L$ is at most 9, then $\text{mincol}_{\mathbb{Z}}(L) = 4$, i.e., $L$ can be colored by four colors.
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**Question**

How many colors are enough to color?
Example

Simple $Z$-coloring
$Z$-coloring by 5 colors
Let $L$ be a $\mathbb{Z}$-colorable link, $\gamma$ a $\mathbb{Z}$-coloring on a diagram $D$ of $L$.

**Simple $\mathbb{Z}$-coloring**

We call $\gamma$ a simple $\mathbb{Z}$-coloring if $\exists d \in \mathbb{N}$ such that for all the crossings in $D$, the differences between the colors of the over arcs and the under arcs are 0 or $d$. 

![Diagram of a link with colors 0, 1, 2, 3, 4, 2, 1, 0, 1, 2, 3, 4]
Theorem 2

Let $L$ be a non-splittable $\mathbb{Z}$-colorable link. If there exists a simple $\mathbb{Z}$-coloring on a diagram of $L$, then $\text{mincol}_\mathbb{Z}(L) = 4$. 
**Theorem 3**

If a non-splittable link $L$ admits a $\mathbb{Z}$-coloring $\gamma$ such that $\#\text{Im}(\gamma) = 5$, then $\text{mincol}_\mathbb{Z}(L) = 4$.

**Proposition**

If a $\mathbb{Z}$-coloring $\gamma$ satisfies $\#\text{Im}(\gamma) = 5$ and $\text{min}\text{Im}(\gamma) = 0$, then $\text{Im}(\gamma) = \{0, 1, 2, 3, 4\}, \{0, 1, 2, 3, 5\}, \{0, 1, 2, 3, 6\}, \{0, 1, 2, 4, 7\}, \{0, 2, 3, 4, 5\}, \{0, 3, 4, 5, 6\}$ or $\{0, 3, 5, 6, 7\}$, up to scale.
In the case $\text{Im}(\gamma) = \{0, 1, 2, 3, 5\}$
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In the case $\text{Im}(\gamma) = \{0, 1, 2, 4, 7\}$
Question

For any non-splittable $\mathbb{Z}$-colorable link $L$, $\text{mincol}_\mathbb{Z}(L) = 4$?

Question

Can any non-splittable $\mathbb{Z}$-colorable link admit a simple $\mathbb{Z}$-coloring?
Thank you for your attention.