On the maximal number of exceptional surgeries

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1. Back grounds

3-dimensional manifold (3-manifold)
A topological space, which locally looks like 3-dimensional Euclidean space.

Example: Our Universe

See for instance;

George, F. R. Ellis,
Cosmology: The shape of the Universe.
Classification of 3-manifolds

Every closed orientable 3-manifold is;

- **Reducible** (containing essential 2-sphere),
- **Toroidal** (containing essential torus),
- **Seifert fibered** (foliated by circles), or
- **Hyperbolic** (admitting Riem. metric of curv. $-1$).

Conjectured by Thurston, (late ’70s)
“Established” by Perelman (2002-03)
(including famous Poincarè Conjecture)
What’s the NEXT?

• Attack the remaining Open Problems.
  (e.g., Virtually Haken Conjecture,
   “Heegaard genus VS rank of $\pi_1$” problem, etc. . . .)

• Relate Geometric & Topological invariants.
  (e.g., Volume conjecture (for knots), etc . . .)

• Study the Relationships between 3-manifolds.
  (e.g., degree one map, Dehn surgery, etc . . .)
  (⇑ Today!)
Let $M$ be a closed orientable 3-manifold and $K$ a knot in $M$.

**Dehn surgery**
1) Remove a neighborhood of $K$ from $M$,
2) Gluing a solid torus back (along slope $\gamma$)
Surgery slope:
Dehn surgery on a knot $K$ is determined by slope $\gamma$ (i.e., isotopy class of simple closed curve) on the peripheral torus $T$ of $K$;

where $\gamma = [f(\text{meridian of } V)]$
Thm. [Wallace ('60), Lickorish ('62)]

Every pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries.

This gives “Network” on the set of 3-manifolds.
§3. Exceptional Surgery

**Theorem (Kawauchi)**

Every pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries on hyperbolic knots.

**Definition (hyperbolic knot)**

A knot $K$ in a 3-manifold $M$ is called hyperbolic if the complement $M - K$ is a Hyperbolic manifold.

⇒ we get “Hyperbolic surgery Network of 3-mfds”
Local picture of Hyp. Surgery Network of 3-mfds

Hyperbolic Dehn Surgery Thm (Thurston)
On a fixed hyperbolic knot, only finitely many Dehn surgeries yield non-hyperbolic 3-manifolds.

NON-hyperbolic (only finitely many) called exceptional surgery for a fixed hyperbolic knot

Hyperbolic (all others)
Question

How many such exceptional surgeries can occur?

Conjecture (Gordon); [Universal bound]

There are at most 10 exceptional surgeries on any hyperbolic knot.

Thm

Agol, Geom. Topol. ('00)
Lackenby, Invent. Math ('00)

There are at most 12 exceptional surgeries on any hyperbolic knot.
§4. Results

The distance $\Delta(\gamma_1, \gamma_2)$ between two slopes $\gamma_1, \gamma_2$ is given by the minimal intersection number of the representatives of the slopes.

**Theorem 1.** [I., to appear JKTR]

Let $\gamma$ be any slope for a hyperbolic knot $K$. Then exceptional surgeries on $K$ along slope $\gamma'$ with $\Delta(\gamma, \gamma') \leq 1$ are at most 10.
When the knot is in the 3-sphere $S^3$, one can parametrize slopes by irreducible fractions. i.e.,

$$\{\text{slope on } T\} \leftrightarrow \mathbb{Q} \cup \{1/0\}$$

Then a slope $\gamma$ is called integral if it corresponds to an integer, i.e., $\Delta(\gamma, [\text{meridian}]) = 1$.

**Corollary 2.** On any hyperbolic knot in $S^3$, there are at most 9 integral exceptional surgeries.
Theorem 3.
On a hyperbolic alternating knot in $S^3$, all non-trivial exceptional surgeries are integral.

Theorem 4.
On a hyperbolic alternating knot in $S^3$, there are at most 10 exceptional surgeries.

Thus Gordon’s Conj. is true for alt.knots.
§5. Results (2)

What’s happen for non-alternating knots?

Example; \([-2, 3, 7]\)-pretzel knot \(K = P(-2, 3, 7)\]

\(K\) admits 7 exceptional surgeries;

\[\varepsilon(K) = \left\{\frac{1}{0}, \frac{16}{1}, \frac{17}{1}, \frac{18}{1}, \frac{37}{2}, \frac{19}{1}, \frac{20}{1}\right\}\]

We cannot apply Theorem 1.

\[\Rightarrow \text{Can we say anything?}\]
In this case...

The slope 18 satisfies;

\[ \Delta (18, \gamma) \leq 2 \quad \text{for } \forall \gamma \text{ in } \varepsilon(K) \]

Because;

- \( K \) is a fibered knot, and so,
  
  the exterior contains an essential lamination \( \mathcal{L} \),

- 18 is a degeneracy slope for \( \mathcal{L} \) [Gabai, Wu].
In general, we have;

**Proposition 5.**

Let \( \varepsilon(K) \) be the set of exceptional surgery slopes for a hyperbolic knot \( K \).

Then there always exists a slope \( \gamma \) such that
\[
\Delta(\gamma, \gamma') \leq 2
\]
holds for any \( \gamma' \in \varepsilon(K) \).

This follows from the existence of degeneracy slope by Gabai-Mosher’s unpublished work.
Here we give an alternative simple proof.

Let $K$ be a hyperbolic knot in a 3-manifold $M$. Take the maximal horotorus $T$ in $M − K$, and let $\gamma$ be the shortest slope on $T$. Then;

**Theorem (C.Adams, 2002 & preprint)**

With only 2 exceptions, the length of $\gamma \geq \sqrt[4]{2}$.

Combining other known results, we see that Prop.5 holds on these 2 exceptions.
Then, Prop. 5 follows from the following:

**Theorem (C. Adams, 1987)**

Suppose that the length of $\gamma \geq \frac{4\sqrt{2}}{}$.

If $\Delta(\gamma, \gamma') \geq 3$, then the length of $\gamma' > 6$.

**Theorem (Agol, Lackenby, 2000)**

Dehn surgery along a slope of length $> 6$ cannot be an exceptional.
Corollary to Proposition 5.
Let $\varepsilon(K)$ be the set of exceptional surgery slopes for a hyperbolic knot $K$. Set;

\[
\#\varepsilon(K) = \text{the number of elements}
\]

\[
\Delta\varepsilon(K) = \max\{\Delta(\gamma, \gamma') | \gamma, \gamma' \in \varepsilon(K)\}
\]

( the diameter of $\varepsilon(K)$ )

Then we have;

**Corollary 6.**
\[
\Delta\varepsilon(K) \leq 8 \text{ implies } \#\varepsilon(K) \leq 10.
\]
Efficiency of Corollary 6.

For example; consider

\[ S = \left\{ \begin{array}{c}
\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{3}, \frac{5}{4}, \frac{7}{4}, \frac{7}{5}, \frac{8}{5} \end{array} \right\} \]

Then \( \Delta S = 8 \) and \( \#S = 12 \).

Thus \( S \) cannot be realized as the set of exceptional surgery slopes for any hyperbolic knot in \( S^3 \).