Distances between boundary slopes of immersed essential surfaces

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§1. Introduction

Let $M$ be a cpt ori 3-mfd with $\partial M \cong T^2$.

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**Def. (immersed essential surface)**

$F$ is an immersed surface in $M$

$\iff$  \exists compact surf $S$ with $\partial S \neq \emptyset$

\exists proper immersion $f : S \hookrightarrow M$

s.t. $F = f(S)$, $f|_{\partial S}$ is an embedding.

$F$ is essential

$\iff f$ is $\pi_1$-injective & $\partial$-$\pi_1$-injective.
Plenty of immersed ess surf’s slope $\overset{\text{def}}{\leftrightarrow}$ isotopy class of s.c.c. on $\partial M$.

$\partial$-slope of immersed surf $F$ $\overset{\text{def}}{\leftrightarrow}$ slope determined by $\partial F$.

Fact (Maher) For a two-bridge knot exterior, all slopes are $\partial$-slopes of immersed ess surf.

Question

Can we have any estimation of $\partial$-slopes of immersed ess surf?
§2. Distances between ∂-slopes

Def (Distance between slopes)

Let \( r_1, r_2 \) be two slopes on \( ∂M \).

Distance \( Δ(r_1, r_2) \) \(\overset{\text{def}}{\leftrightarrow}\) minimal geometric intersection number of their representative.

Notation:

\[ \chi(F) := \text{Euler characteristic of } S. \]
\[ ∂F := f(∂S). \]
\[ #∂F := \text{number of conn compo of } ∂F. \]
Fact (Hass-Rubinstein-Wang)

Let $M$ be a cpt ori 3-mfd with $\partial M \cong T^2$ s.t. $\text{int}M$ is complete hyperbolic. Let $F_i$ be immersed essential surf in $M$ with $\partial$-slope $r_i$ for $i = 1, 2$.

$\Rightarrow \quad \Delta(r_1, r_2) < \frac{43}{4} \cdot \frac{-\chi(F_1)}{\#\partial F_1} \cdot \frac{-\chi(F_2)}{\#\partial F_2}$

In particular, if $F_i$’s are orientable,

$\Delta(r_1, r_2) < \frac{43}{4} \cdot \text{genus}(F_1) \cdot \text{genus}(F_2)$
§3. Results

Let $F_1, F_2$ be immersed essential surf in $M$ with $\partial$-slopes $r_1, r_2$.

**Theorem 1.**

If $M$ is Seifert fibered, then

$$\Delta(r_1, r_2) \leq 2 \left( \frac{-\chi(F_1)}{\#\partial F_1} + \frac{-\chi(F_2)}{\#\partial F_2} \right) + 4$$

In particular, if $F_i$'s are orientable,

$$\Delta(r_1, r_2) \leq 4(\text{genus}(F_1) + \text{genus}(F_2))$$
Theorem 2.

If int$M$ is hyperbolic and $r_i$’s are integral w.r.t. some meri-longi system on $\partial M$,

$$\Delta(r_1, r_2) < 6 \left( \frac{-\chi(F_1)}{\#\partial F_1} + \frac{-\chi(F_2)}{\#\partial F_2} \right)$$

In particular, if $F_i$’s are orientable,

$$\Delta(r_1, r_2) < 12(\text{genus}(F_1) + \text{genus}(F_2) - 1)$$

Compare to (H-R-W)’s

$$\Delta(r_1, r_2) < \frac{43}{4} \cdot \text{genus}(F_1) \cdot \text{genus}(F_2)$$
Theorem 3.

If $M$ is a knot exterior in $S^3$ and $F_i$’s are immersed spanning surface without triple points, then

$$\Delta(r_1, r_2) \leq 2 \left( \frac{-\chi(F_1)}{\# \partial F_1} + \frac{-\chi(F_2)}{\# \partial F_2} \right) + 4$$
Example

\( K = (-2, 3, n) \)-pretzel knot

\( n = 7, 9, 11, \ldots \)

\( \exists F_i: \) embedded essential surf

with \( \partial \)-slope \( r_i \) for \( i = 1, 2 \) s.t.

\[
\begin{array}{cccccc}
-\chi(F_1) & \#\partial F_1 & r_1 & -\chi(F_2) & \#\partial F_2 & r_2 \\
n - 6 & 1 & 16 & n - 5 & 2 & \frac{n^2 - n - 5}{(n - 3)/2} \\
\end{array}
\]

Then \( \Delta(r_1, r_2) = n^2 + 7n - 29; \)

quadratic with respect to genera
Theorem 4.

If $M$ is a small Seifert fibered space, then

$$\left(\frac{-\chi(F_1)}{\#\partial F_1} + \frac{-\chi(F_2)}{\#\partial F_2}\right) + 2 \leq \Delta(r_1, r_2)$$
Let $M$ be a cpt ori 3-mfd with $\partial M \cong T^2$ s.t. int$M$ is complete hyperbolic. If two slopes $r_1, r_2$ are both integral slopes w.r.t. some meri-longi system on $\partial M$, and both $r_1$-, $r_2$-surgeries yield non-hyperbolike manifolds, then $\Delta(r_1, r_2) \leq 8$. Thus there are at most NINE such integral non-hyperbolike surgeries.