Hyperbolicity of sections in surface bundles

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1. INTRODUCTION

$K$ : knot in 3-mfd. $M$

$K$ is **Hyperbolic** if $M - K$ admits a hyperbolic structure.

**Question**

How can we recognize whether a given knot is hyperbolic or not?

e.g., Using **DIAGRAM**

If $K$ in $S^3$ has alternating diagram, then $K$ is hyperbolic.
In this talk, we consider:

\( M_f: \text{surface bundle}, \) i.e.,
\[ F \times [0, 1]/\{(x, 0) = (f(x), 1)\} \]

\( F: \) closed orientable surface of genus greater than 1.

\( f: F \rightarrow F, \) orientation preserving homeo.

\( K: \) knot appearing as section in \( M_f \)
2. RESULTS

\[ M_f \begin{cases} 
\text{Hyperbolic,} \\
\text{Small Seifert fibered, or} \\
\text{Toroidal}
\end{cases} \]

(i) \( M_f \): Hyperbolic

\begin{theorem}
In hyperbolic \( M_f \), every knot appearing as a section is hyperbolic.
\end{theorem}

Remark \( M_f \): hyperbolic

iff \( f \) isotopic pseudo-Anosov map
(ii) $M_f$: Small Seifert fibered

We may assume;

$f$ is Periodic
i.e., $f^n = \text{identity}$ for some $n$

and

$f$ is Irreducible, i.e.,
no invariant essential 1-submfd
(set of mutually non-parallel, non-trivial curves).
(ii) $f$ has no fixed points

THEOREM 2
If $f$ is periodic, irreducible, has no fixed points, then every knot in $M_f$ appearing as a section is hyperbolic.
(iib) \( f \) has **Fixed points**

Fix a fixed point \( x_0 \) of \( f \) on \( F \).

\( K \): knot appearing as section

We may assume:

\( K \) runs through \( x_0 \times \{0\} \).

\( p : M_f \to F \) natural projection
Let $\alpha_K \in \pi_1(F, x_0)$: an element represented by the projection $p(K)$.

**THEOREM 3**

$K$ is hyperbolic iff

$$\alpha_K f_*(\alpha_K) \cdots f_*^{n-1}(\alpha_K) \neq 1$$

in $\pi_1(F, x_0)$, where $f_*$ is induced isom. of $\pi_1(F, x_0)$ and $n$ is the period of $f$. 
(iii) $M_f$: Toroidal

(iiiia) $M_f$: Toroidal Seifert

We may assume; $f$ is Periodic

**Assumption:** $f$ has fixed pt $x_0$

**THEOREM 4**

$K$ is hyperbolic if and only if

$$\alpha_K f_*(\alpha_K) \cdots f_{n-1}^*(\alpha_K)$$

is filling.

\[ \beta \text{ is filling if } \]

\[ F - c = \text{disks} \]

for any $c$

with $[c] = \beta$
(iiiib) $M_f$: Toroidal, not Seifert

We may assume that;

there exists $C_f = \{C_1, C_2, \ldots\}$; set of $f$-invariant essential 1-submfd's.

Assumption

$C_f$ is a finite set,
and, $f$ has a fixed point $x_0$
Precisely:

\[ C_f = \{C_1, \ldots, C_n\}: \]

a system of essential 1-submfds such that

\[ f(C_i) = C_i, \quad f(N(C_i)) = N(C_i) \]

for some small nbd \( N(C_i) \), and any other isotopically \( f \)-invariant essential 1-submfd can be isotoped into \( N(C_i) \) for some \( i \).
THEOREM 5

Let $c$: representative of $\alpha_K$ intersecting $C_f$ minimally.

If $c$ intersects all $C_i$'s and is well-terminated w.r.t. $C_f$, then $K$ is hyperbolic.
A representative $c$ of $\alpha_K$ is well-terminated if

$$f(\tau_i) \text{ is no homotopic to } \tau'_i \text{ rel } \{x_0\} \cup C_i$$
c.f. (Special case of Toroidal Seifert)

$M_f$: trivial surface bundle, i.e., $M_f = F \times S^1$

$K$: knot in $M_f$ appearing as section

$\alpha_K$: element of $\pi_1(F,x_0)$ represented by $p(K)$

**THEOREM (Kra)**

$K$ is hyperbolic if and only if $\alpha_K$ is filling.