Bounds on exceptional surgery slopes

Kazuhiro Ichihara
Nara University of Education

including a joint work with
Kimihiko Motegi and Hyun-Jong Song
§0. Back grounds

3-dimensional manifold (3-manifold)

A topological space, which locally looks like 3-dimensional Euclidean space.

Example: Our Universe
Classification of 3-manifolds

Every closed orientable 3-manifold is:

- Reducible (containing essential 2-sphere),
- Toroidal (containing essential torus),
- Seifert fibered (foliated by circles), or
- Hyperbolic (admitting Riem. metric of curv. $-1$).

including famous Poincaré Conjecture (1904)
conjectured by Thurston (late ’70s)
established by Perelman (2002-03)

⇒ Let us study the relationships between 3-mfds.
§1. Introduction (Dehn Surgery)

Let $M$ be a closed orientable 3-manifold and $K$ a knot in $M$.

**Dehn surgery**

1) Remove a neighborhood of $K$ from $M$,
2) Gluing a solid torus back (along slope $\gamma$)
Thm. [Wallace ('60), Lickorish ('62)]

Every pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries.

This gives “Network” on the set of 3-manifolds.
Recall: surgery slope

Dehn surgery on a knot \( K \) is determined by slope \( \gamma \) (i.e., isotopy class of simple closed curve) on the peripheral torus \( T \) of \( K \);

where \( \gamma = [ f(\text{meridian of } V) ] \)
Remark
When the knot is in the 3-sphere $S^3$, or $\text{ZH}S^3$ by using a standard meridian-longitude system, one can parametrize slopes by irreducible fractions.

i.e., \[ \{ \text{slope} \} \overset{1:1}{\leftrightarrow} \left\{ \frac{p}{q} \right\} \]

Recall: $\text{ZH}S^3$
closed ori. 3-mfd with the same homology as $S^3$
Hyperbolic Surgery Theorem [Thurston]
Each hyperbolic knot admits only finitely many Dehn surgeries producing non-hyperbolic 3-mfds.

Recall: hyperbolic knot
= knot with hyperbolic complement

Such surgeries are now called exceptional surgeries.

We consider three Conjectures on exceptional surgeries.
§2. Conjectures & Result

Conjecture 1. (Denominator)

Recall: trivial surgery = the surgery along $1/0$

Conjecture 1. [Gordon]
Every non-trivial exceptional surgery slope $p/q$
for a hyperbolic knot in $S^3$ satisfies $|q| \leq 2$.

Known facts: If the obtained manifold is;
- reducible, then $|q| \leq 1$  
  [Gordon-Luecke, 1987]
- toroidal, then $|q| \leq 2$  
  [Gordon-Luecke, 1995]
- spherical, then $|q| \leq 2$  
  [Boyer-Zhang, 1995]
Conjecture 2. (vs genera of knots)

In the following, $g(K)$: the genus of a knot $K$. (i.e., minimal genus of Seifert surfaces for $K$)

**Conjecture 2. [Teragaito]**
Every non-trivial exceptional surgery slope $p/q$ for a hyperbolic knot in $S^3$ satisfies $|p/q| \leq 4g(K)$.

**Known facts**: If the obtained mfd is;
- non-hyperbolic, $|p/q| \leq 10.05 \ g(K)$  [I., 2001]
- including Klein bottle, $|p/q| \leq 4g(K)$  [I.-Teragaito, 2003]
- a lens space, $|p/q| \leq 4g(K) + 3$  [Rassmussen, 2004]
Theorem. [I.]

Let $p/q$ be a non-trivial exceptional surgery slope for a hyperbolic knot $K$ in $\mathbb{Z}HS^3$.

Then at least one of the following always holds:

(i) $|q| \leq 2$

(ii) $|p/q - R_F| \leq 4g(F)$ for $\forall$ essential surface $F$,
    in particular, $|p/q| \leq 4g(K)$.

Therefore, for each hyperbolic knot in $S^3$, at least one of Conjectures 1 or 2 must be true.
Terminologies

Let $E(K)$ denote the exterior of a knot $K$ in a 3-manifold $M$ (i.e., $M$—(open tubular neighborhood of $K$))

For an embedded surface $F$ in $E(K)$ with $\partial F \neq \emptyset$, (possibly non-orientable)

we call $F$ essential if $F$ is incompressible & $\partial$-incompressible, (e.g., minimal genus Seifert surface for a knot)

$\partial$-slope of $F$ means the slope on $\partial E(K)$ determined by $\partial F$, (we denote it by $R_F$)

$g(F) := (-\chi(F) - \#\partial F + 2)/2$. (when $F$ is orientable, it means the usual genus)
[Sketch of Proof]

$K$ : a hyperbolic knot in $\mathbb{Z}H S^3$

\textbf{Fact. [Gabai-Mosher (unpublished)]}

\exists \text{very full essential lamination } \mathcal{L} \text{ in } E(K).

A \text{ lamination} (i.e., a codim. one foliation on a closed subset) is called \textbf{essential} if it has

\begin{itemize}
  \item no sphere leaf,
  \item no torus leaf bounding a solid torus,
  \item irreducible complementary regions with incomp. boundary,
  \item no compressing monogons.
\end{itemize}
An essential lamination is called very full if every complementary region is an ideal polygonal bundle.

For such an essential lamination $\mathcal{L}$, we can find an annulus $A$ connecting a leaf of $\mathcal{L}$ to $\partial E(K)$.

One curve of the boundary $\partial A$ determines a slope $d_{\mathcal{L}}$ on $\partial E(K)$, which we call degeneracy slope.
Case (i): $d_\mathcal{L} = 1/0$

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**Fact. [Wu, 1998]**

For degeneracy slope $\delta_\mathcal{L}$ & exceptional surgery slope $p/q$, the distance $\Delta(p/q, \delta_\mathcal{L}) \leq 2$.

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Recall:

The distance $\Delta(\gamma, \gamma')$ between slopes $\gamma, \gamma'$ is the minimal geometric intersection number of the representatives of $\gamma, \gamma'$. For slopes on $\partial E(K)$, $\Delta(a/b, c/d) = |ad - bc|$.

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Thus, if $d_\mathcal{L} = 1/0$, we have $|q| = \Delta(p/q, 1/0) \leq 2$. 
Case (ii) : $d_L \neq 1/0$

In this case, we have

$$|p/q - R_F| \leq |p/q - d_L| + |d_L - R_F| \leq 2 + |d_L - R_F|$$

Proposition 1.

We have $\Delta(d_L, R_F) \leq 4g(F) - 2$.

Remark: [Gabai] already showed when $F$ is a Seifert surface.

Since $\Delta(a/b, c/d) = |ad - bc| \geq |a/b - c/d|$, we have

$$|p/q - R_F| \leq 2 + |d_L - R_F| \leq 2 + 4g(F) - 2 = 4g(F) \quad \square$$
Remark

(i) A similar result of Fact [Wu] for essential surface was obtained by [Boyer-Gordon-Zhang, 2001].

(ii) A similar result of Proposition 1 for essential surface was obtained by [I.-Ozawa, 2002], which motivated this study.
§3. Another conjecture & Examples

**Conjecture 3. [Goda-Teragaito]**
If Dehn surgery on a hyperbolic knot $K$ in $S^3$ along slope $r$ produces a lens space, then

\[ 2g(K) + 8 \leq |r| \leq 4g(K) - 1 \]

Based on Proposition 1, together with known facts, we have a **Condition** such that, toward Conj 3, we only consider the knots satisfying it.
Suppose that Dehn surgery on a hyperbolic knot $K$ in $S^3$ along slope $r$ produces a lens space.

(spherical manifold, in general)

**Proposition 2.**

If $|r| > 4g(K) - 1$, then $E(K)$ admits a very full lamination with meridional degeneracy slope which have unique essential annulus connecting a leaf of $\mathcal{L}$ to $\partial E(K)$.

**Problem**: Does there exist such a knot in $S^3$?
To find Counter-example...

**Observation**

A knot $K$ in $S^3$ satisfies the conditions in Prop. 2 if $K$ is hyperbolic and fibered, and Dehn surgery on $K$ along the longitudinal slope gives a non-hyperbolic manifold.

Because, when a knot $K$ is hyperbolic and fibered, we have the essential lamination which appears as a suspension of the invariant foliation for the monodromy of $E(K)$. 

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Example 1 [Gabai]

[Gabai] (together with Kazez) first found that the knot $8_{20}$ satisfies the conditions in Prop 2. Generalizing $8_{20}$, we see that the pretzel knots $P(2, n, -n)$ with $n \geq 3$:odd also satisfies the conditions in Prop 2.

However, these do not give counterexamples, by virtue of [Gordon, 1999].
Example 2

**Theorem [I.-Motegi-Song, 2008]**

∃ infinitely many small, fibered hyperbolic knots $K$ in $S^3$ on each of which Dehn surgery along the longitudinal slope produces a Seifert fiber space.

Thus, by Lemma,

such knots are all satisfies the conditions in Prop 2.

However, these also do not give counterexamples, by recent preprint [Lackenby-Meyerhoff]...
Byproducts of Proposition 1.

(i) Any degeneracy slope for a very full essential lamination in a hyperbolic alternating knot exterior is meridional.
(a part of the conjecture by [Gabai-Kazez])

(ii) We obtain two bounds about $\partial$-slopes for a hyperbolic knot in $\mathbb{Z}HS^3$, at least one of which always holds:
This gives a generalization to the result on Montesinos knots obtained by [I.-Mizushima].