Heegaard gradient of Seifert fibered 3-manifold

Kazuhiro Ichihara

Nara Women’s Univ.
JSPS Research Fellow.

Nara 奈良女子大学 学振特別研究員（PD）
§1. Introduction

§2. Heegaard gradient

§3. Vanishing of HG

Motivation
Marc Lackenby,
Heegaard splittings,
the virtually Haken conjecture
and Property (τ).

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Notation
$M$ denotes a closed, orientable, irreducible 3-manifold.
1. Introduction

**Virtually Haken Conjecture**

If $\pi_1(M)$ is infinite, then $M$ is virtually Haken.

An irr. 3-mfd. $M$ is

- **Haken** $\iff$ $M \ni$ an incompressible surface,
- **virtually Haken** $\iff$ $\exists \widetilde{M}$, finite cover s.t. $\widetilde{M}$ is Haken.

If VHC is true

- Hyperbolization Conj. and
- Topologically rigid Conj.

is true.
Theorem [Casson-Gordon]

If $M$ has an irreducible & weakly reducible Heegaard splitting,
then $M$ is Haken.

$\chi^h(M)$ : the min. of
$\{ -\chi(F) \mid F \text{ is a Heegaard surf. of } M \}$. 
Proposition [Lackenby]

\( \tilde{M} : \) a \( d \)-fold cover of \( M \).

If \( d \) is ‘sufficiently’ large &

\[ \frac{\chi^h_-(\tilde{M})}{d} = \chi^h_-(M), \]

then \( \tilde{M} \) is Haken,

i.e. \( M \) is virtually Haken.
§2. Heegaard gradient

**Definition**

\[ \mathcal{M} := \{ M_i \to M, \ d_i\text{-fold cover of } M \} \]

the **Heegaard gradient of** \( \mathcal{M} \) **is**

\[ HG(\mathcal{M}) = \inf_i \frac{\chi^h(M_i)}{d_i} . \]

**Corollary**

\[ \exists \mathcal{M} = \{ M_i \to M \} : \]

an infinite set of finite covers

s.t. \( HG(\mathcal{M}) = \chi^h(M) \)

\[ \Rightarrow M \text{ is virtually Haken.} \]
Theorem [Lackenby]

\[ \exists \mathcal{M} : \text{an infinite lattice} \]

of finite \textbf{regular} covers of \( M \)

s.t. \( \text{HG}(\mathcal{M}) > 0 \) \&

\( \pi_1(M) \) does not have \textbf{Property(\( \tau \))} w.r.t. \( \mathcal{M} \),

\[ \Rightarrow M \text{ is virtually Haken.} \]

A set \( \mathcal{M} = \{ M_i \to M \} \) of finite covers is a \textbf{lattice}

def.
**Theorem [I.]**

\( M : \text{ non-Haken} \quad \& \quad \chi_h(M) > 0 \)

\( \exists M = \{ M_i \to M \} : \) 

an infinite set of finite covers 

s.t. \( HG(M) > 0 \)

\( \Rightarrow M \) is virtually Haken.
§3. Vanishing of HG

Problem

For which \( M \) and \( \mathcal{M} \),

\[
HG(\mathcal{M}) = 0
\]

Fact

\( M \) : a surface bundle over \( S^1 \).

\( \Rightarrow \) \( HG(\mathcal{M}) = 0 \) for

- a set \( \mathcal{M} \) of cyclic covers

- in the \( S^1 \)-direction.

\( \Rightarrow \) \( HG(\bar{\mathcal{M}}) = 0 \) for

- the set \( \bar{\mathcal{M}} \) of all finite covers.
Definition

Heegaard gradient $HG(M)$ of $M$

$= HG(\tilde{M})$ for the set $\tilde{M}$

of all finite covers of $M$.

Corollary

$M$ : a virtual surface bundle,
i.e. $M$ is finitely covered
by a surface bundle over $S^1$,
$\Rightarrow HG(M) = 0$.

Proposition[Lackenby]

$M$ : a reducible 3-manifold.

$HG(M) = 0$

$\iff M$ is a virtual surface bundle,
$\iff M \cong S^2 \times S^1$ or $RP^3 \# RP^3$. 
Question

\( M \) is a virtual surface bundle

\( \iff \) \( HG(M) = 0 \)

Ans. is NO!

Theorem[I.]

\( M \) : a Seifert fibered 3-mfd.

\( HG(M) = 0 \)

\( \iff \) \( M \) : virtual surface bundle

or \( M \) : virtual circle bundle

over a surf. with \( \chi < 0 \).

\( \iff \) \( |\pi_1(M)| = \infty \).
**Corollary**

\( M : \) a Seifert fibered 3-mfd.

\[ \Rightarrow \HG(M) \leq 0. \]

**Theorem [I.]**

\( M : \) a Seifert fibered 3-mfd.

\( \mathcal{M} = \{ M_i \to M \} : \)

a set of its finite covers.

\[ \HG(\mathcal{M}) = 0 \]

\[ \iff \] the set of vertical degrees of \( \{ M_i \to M \} \) is unbounded.
**Example**

$F$: a closed surface with $\chi < 0$

$M^e_\chi$: a circle bundle over $F$

with $\chi > 0 \& e(M) = e$.

**Proposition**

$\exists M = \{ M^1_{2i\chi} \to M^1_\chi \}$ s.t. $HG(M) = 0$.

**Remark**

$\forall M^1_\chi$ is Haken, but its irreducible Heegaard splitting is strongly irreducible.