On exceptional surgeries on Montesinos knots

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joint works with

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This talk is based on

- K. Ichihara and I.D. Jong
  *Cyclic and finite surgeries on Montesinos knots*

- K. Ichihara and I.D. Jong
  *Toroidal Seifert fibered surgeries on Montesinos knots*
  Preprint, arXiv:1003.3517

- K. Ichihara, I.D. Jong and S. Mizushima
  *Seifert fibered surgeries on alternating Montesinos knots*
  in preparation.
1. Introduction
As a consequence of the Geometrization Conjecture including famous Poincaré Conjecture (1904) conjectured by Thurston (late ’70s) established by Perelman (2002-03),
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Classification of 3-manifolds

As a consequence of the **Geometrization Conjecture**

including famous **Poincaré Conjecture** (1904)

conjectured by Thurston (late ’70s)

established by Perelman (2002-03),

every closed orientable **3-manifold** is;

- **Reducible** (containing essential 2-sphere),
- **Toroidal** (containing essential torus),
- **Seifert fibered** (foliated by circles), or
- **Hyperbolic** ($\exists$ Riem. metric of curv. $-1$).
• Attack the remaining **Open Problems**.  
  (e.g., Virtually Haken Conjecture . . . )
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• Relate **Geometric & Topological** invariants
  (e.g., Volume conjecture . . .)
What’s the NEXT?

• Attack the remaining Open Problems.
  (e.g., Virtually Haken Conjecture . . .)

• Relate Geometric & Topological invariants
  (e.g., Volume conjecture . . .)

• Study the Relationships between 3-mfds.
  (e.g., Dehn surgery . . .)
  (⇑ Today!)
Dehn surgery on a knot

- $K$: a knot in a 3-mfd $M$
- $E(K)$: the exterior of $K$ ($:= M - (\text{open nbd. of } K)$)
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Dehn surgery on a knot

- $K$: a knot in a 3-mfd $M$
- $E(K)$: the exterior of $K$ ($:= M - (\text{open nbd. of } K)$)

Dehn surgery: Gluing a solid torus to $E(K)$

$$\gamma = \left[ f(m) \right] : \text{surgery slope}$$
Dehn surgery on a knot

- $K$: a knot in a 3-mfd $M$
- $E(K)$: the exterior of $K$ ($:= M - (\text{open nbd. of } K)$)

**Dehn surgery**: Gluing a solid torus to $E(K)$

\[ \gamma = [f(m)]: \text{surgery slope} \]

**Theorem** [Lickorish (1962), Wallace (1960)]

Every pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries.
Exceptional surgery

Dehn surgery on a hyperbolic knot (i.e., knot with hyperbolic complement) yielding a non-hyperbolic mfd.

Theorem [Thurston (1978)]

Exceptional surgeries are only finitely many for each hyperbolic knot.
Exceptional surgery

Dehn surgery on a **hyperbolic** knot (i.e., knot with hyperbolic complement) yielding a non-hyperbolic mfd.

Theorem [Thurston (1978)]

Exceptional surgeries are **only finitely many** for each hyperbolic knot.

Each exceptional surgery is either:

- **Reducible** surgery  (yielding a mfd. containing an essential $S^2$)
- **Toroidal** surgery  (yielding a mfd. containing an essential $T^2$)
- **Seifert** surgery   (yielding a Seifert fibered mfd.)
Montesinos knot $M(R_1, \ldots, R_l)$ in $S^3$

A knot admitting a diagram obtained by putting rational tangles $R_1, \ldots, R_l$ together in a circle.

arcs on a 4-punctured sphere, and $\frac{1}{2}$-tangle

**length** of the knot

$= \text{minimal number of rational tangles}$

$M(\frac{1}{2}, \frac{1}{3}, -\frac{2}{3}) \uparrow$
Montesinos knot $M(R_1, \ldots, R_l)$ in $S^3$

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$P(a_1, \cdots, a_n) = M\left(\frac{1}{a_1}, \cdots, \frac{1}{a_n}\right) : (a_1, \cdots, a_n)$-pretzel knot.
Problem

Classify all the exceptional surgeries on hyperbolic Montesinos knots.
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Classify all the exceptional surgeries on hyperbolic Montesinos knots.

Remark [Menasco], [Oertel], [Bonahon-Siebenmann]

Non-hyperbolic Montesinos knots are
\[ T(2, n), \quad P(-2, 3, 3)(=T(3, 4)), \quad P(-2, 3, 5)(=T(3, 5)). \]

\[ T(x, y) : \] the \((x, y)\)-torus knot.
Problem

Classify all the exceptional surgeries on hyperbolic Montesinos knots.

Remark [Menasco], [Oertel], [Bonahon-Siebenmann]

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\[ T(x, y) \]: the \((x, y)\)-torus knot.

Remark

Dehn surgeries on the torus knots have been completely classified by Moser (1971).
Known facts: Length other than 3

\( K \): hyperbolic Montesinos knot with length \( l \)
Known facts: Length other than 3

\[ K : \text{hyperbolic Montesinos knot with length } l \]

- \( l \leq 2 \Rightarrow K \) is a two-bridge knot.
  Exceptional surgeries for them are \textit{completely classified} [Brittenham-Wu (1995)].
Known facts: Length other than 3

$K$: hyperbolic Montesinos knot with length $l$

- $l \leq 2 \Rightarrow K$ is a two-bridge knot. Exceptional surgeries for them are completely classified [Brittenham-Wu (1995)].

- $l \geq 4 \Rightarrow K$ admits no exceptional surgery [Wu (1996)].
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\[ K : \text{hyperbolic Montesinos knot with length } l \]

- \[ l \leq 2 \Rightarrow K \text{ is a two-bridge knot.} \]
  Exceptional surgeries for them are completely classified [Brittenham-Wu (1995)].

- \[ l \geq 4 \Rightarrow K \text{ admits no exceptional surgery [Wu (1996)].} \]

Remains

Exceptional surgeries on \( M(R_1, R_2, R_3) \) (i.e. \( l = 3 \))
\item \textit{No} reducible surgeries on Montesinos knots [Wu (1996)].
Known facts : Reducible / Toroidal surgery

• \( \not\exists \) reducible surgeries on Montesinos knots [Wu (1996)].

• Toroidal surgeries on Montesinos knots are completely classified [Wu (2006)].
Known facts: Reducible / Toroidal surgery

- **No reducible** surgeries on Montesinos knots [Wu (1996)].

- **Toroidal** surgeries on Montesinos knots are completely classified [Wu (2006)].

Remains

Seifert surgeries on $M(R_1, R_2, R_3)$
2. Toroidal Seifert surgery
Known facts: Toroidal Seifert surgery

Recall: Each exceptional surgery is either:

- Reducible (conjectured: $\not\exists$ (Cabling Conjecture)),
- Toroidal,
- Seifert.
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Remark [Eudave-Muñoz (2002)]

They are not exclusive.

(i.e., there are non-empty intersection)
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Theorem [Motegi (2003)]

A knot $K$ with $|\text{Sym}^*(K)| > 2$ admits no toroidal Seifert surgery.
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They are not exclusive.

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Theorem [Motegi (2003)]

A knot \( K \) with \( |\text{Sym}^*(K)| > 2 \) admits no toroidal Seifert surgery.

In particular, other than the trefoil knot, no two-bridge knots admit toroidal Seifert surgeries.
Theorem [I.-Jong]

Montesinos knots admit no toroidal Seifert surgeries other than the trefoil knot.
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**Montesinos knots** admit no **toroidal Seifert surgeries** other than the **trefoil knot**.

Corollary

A hyperbolic Montesinos knot admits no toroidal Seifert surgery.
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Corollary

A hyperbolic Montesinos knot admits no toroidal Seifert surgery.

Remains

Atoroidal Seifert surgeries on $M(R_1, R_2, R_3)$ (i.e. yielding a Seifert mfd. over $S^2$ with $\leq 3$ exceptional fibers)
3. Cyclic/Finite surgery
Problem

On (hyperbolic) knots in $S^3$, determine all Dehn surgeries giving 3-mfds with cyclic or finite fundamental groups.
Problem

On (hyperbolic) knots in $S^3$, determine all Dehn surgeries giving 3-mfds with cyclic or finite fundamental groups.

We call such surgeries cyclic surgeries / finite surgeries respectively.

Remark

- Such mfds are all Seifert fibered.
- On non-hyperbolic knots, such surgeries have been classified.
We give a complete classification of cyclic / finite surgeries on Montesinos knots.

**Theorem [I.-Jong (2009)]**

\(K\) : hyperbolic Montesinos knot
\(K(r)\) : the manifold obtained by surgery on \(K\) along the slope \(\gamma\) corresponding to \(r\).

- If \(\pi_1(K(r))\) is cyclic, then \(K = P(-2, 3, 7)\) and \(r = 18\) or \(19\).
- If \(\pi_1(K(r))\) is acyclic finite, then \(K = P(-2, 3, 7)\) and \(r = 17\), or \(K = P(-2, 3, 9)\) and \(r = 22\) or \(23\).
[**Watson**]: for $p \in \{5, 7, \cdots, 25\}$,
Surgery obstructions from Khovanov homology.
(by using Khovanov homology)

[**Futer-Ishikawa-Kabaya-Mattman-Shimokawa**]:
a complete classification of finite surgeries on
($-2, p, q$)-pretzel knots with $p, q$: odd positive.
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[Watson]: for $p \in \{5, 7, \cdots, 25\}$,

Surgery obstructions from Khovanov homology.


(by using Khovanov homology)

[Futer-Ishikawa-Kabaya-Mattman-Shimokawa]:

a complete classification of finite surgeries on

$(-2, p, q)$-pretzel knots with $p, q$: odd positive.


Remains

Atoroidal Seifert surgeries on $K = M(R_1, R_2, R_3)$ with

$|\pi_1(K(r))| = \infty$

(i.e. yielding a Seifert mfd. over $S^2(n_1, n_2, n_3)$ with

$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq 1$)
4. On alternating knots
Alternating knots

**Alternating knot**

An alternating diagram = the crossings alternate under, over, under, over, as you travel along the knot. A knot is alternating if it admits an alternating diagram.

\[ P(3, 5, 8) = \]

\[ P(-3, 5, 8) = \]
**Alternating knots**

**Alternating knot**

An alternating diagram = the crossings alternate under, over, under, over, as you travel along the knot. A knot is alternating if it admits an alternating diagram.

\[ P(3, 5, 8) = \quad P(-3, 5, 8) = \]

\[ \begin{array}{c}
\quad \\
\end{array} \]

**Remark [Lickorish-Thistlethwaite]**

A Montesinos knot is alternating if and only if its reduced Montesinos diagram is alternating. In particular, \( M(R_1, \ldots, R_l) \) is alternating if \( R_1, \ldots, R_l \) have the same sign.
Theorem [I.-Jong-Mizushima]

If $K = M(R_1, R_2, R_3)$ with $R_1, R_2, R_3 > 0$ (i.e. $K$ is alternating) admits an atoroidal Seifert surgery, then $K = P(a, b, c)$ with odd integers $3 \leq a < b < c$. 
Results: atoroidal Seifert surgery

Theorem [I.-Jong-Mizushima]

If $K = M(R_1, R_2, R_3)$ with $R_1, R_2, R_3 > 0$ (i.e. $K$ is alternating) admits an atoroidal Seifert surgery, then $K = P(a, b, c)$ with odd integers $3 \leq a < b < c$.

Theorem [Wu]

If $M\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$ with $q_1 \leq q_2 \leq q_3$ admits an atoroidal Seifert surgery, then $q_1 = 2$, $(q_1, q_2) = (3, 3)$, or $(q_1, q_2, q_3) = (3, 4, 5)$.
Results: atoroidal Seifert surgery

Theorem [I.-Jong-Mizushima]

If \( K = M(R_1, R_2, R_3) \) with \( R_1, R_2, R_3 > 0 \) (i.e. \( K \) is alternating) admits an atoroidal Seifert surgery, then \( K = P(a, b, c) \) with odd integers \( 3 \leq a < b < c \).

Theorem [Wu]

If \( M(p_1/q_1, p_2/q_2, p_3/q_3) \) with \( q_1 \leq q_2 \leq q_3 \) admits an atoroidal Seifert surgery, then \( q_1 = 2, (q_1, q_2) = (3, 3) \), or \( (q_1, q_2, q_3) = (3, 4, 5) \).

Corollary

An alternating hyperbolic Montesinos knot with length 3 admits no Seifert surgery.
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Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then
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(I) $l \leq 2$ (i.e., $K$ is a two-bridge knot)  
    $\Rightarrow$ such surgeries are completely classified.
Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then

\begin{itemize}
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  \\
  \item[(II)] $l = 3$: and,
\end{itemize}
Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then

(I) $l \leq 2$ (i.e., $K$ is a two-bridge knot)  
    $\Rightarrow$ such surgeries are completely classified.

(II) $l = 3$: and,

- $K$ admits no reducible surgery,
Suppose that a hyperbolic Montesinos knot \( K \) admits an exceptional surgery. Then

(I) \( l \leq 2 \) (i.e., \( K \) is a two-bridge knot)
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- toroidal surgeries on \( K \) are classified,
Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then

(I) $l \leq 2$ (i.e., $K$ is a two-bridge knot) \[ \Rightarrow \] such surgeries are completely classified.

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- $K$ admits no toroidal Seifert surgery,
Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then

(I) $l \leq 2$ (i.e., $K$ is a two-bridge knot) 
$\Rightarrow$ such surgeries are completely classified.

(II) $l = 3$: and,

- $K$ admits no reducible surgery,
- toroidal surgeries on $K$ are classified,
- $K$ admits no toroidal Seifert surgery,
- cyclic / finite surgeries on $K$ are classified,
Suppose that a hyperbolic Montesinos knot $K$ admits an exceptional surgery. Then

(I) $l \leq 2$ (i.e., $K$ is a two-bridge knot) \quad \Rightarrow \text{such surgeries are completely classified.}$

(II) $l = 3$: and,

- $K$ admits no reducible surgery,
- toroidal surgeries on $K$ are classified,
- $K$ admits no toroidal Seifert surgery,
- cyclic / finite surgeries on $K$ are classified,
- in addition, if $K$ is alternating, then $K$ admits no Seifert surgery.