Seifert fibered surgeries on Montesinos knots

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§1. Introduction

**Dehn surgery on a knot $K$ in $S^3$**

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2) Gluing a solid torus back (along slope $\gamma$).

$\gamma = [f(m)]$
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\[ \{ \text{slopes} \} \leftrightarrow Q \cup \{ 1/0 \} \]

For \( r \in \mathbb{Q} \),

\textit{r-surgery}: surgery along a slope parameterized by \( r \).

\( K(r) \): the manifold obtained by \( r \)-surgery along a knot \( K \).

Today, we assume that all surgeries are non-trivial. Namely, we set aside the trivial(1/0-) surgery.
Problem

On hyperbolic knots in $S^3$, determine all non-trivial Dehn surgeries producing non-hyperbolic 3-mfds. (Determine exceptional surgeries.)

Such surgeries are only finitely many. [Thurston]

Types of exceptional surgeries

- Reducible surgery
- Toroidal surgery
- Seifert fibered (SF) surgery

(as a consequence of Geometrization conjecture)
Today’s problem
Determine all exceptional surgeries on hyperbolic Montesinos knots.

Montesinos link $M(R_1, \ldots, R_l)$

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Montesinos link $M(R_1, \ldots, R_l)$

$R_1 \quad R_2 \quad \cdots \quad R_l$

$l$: length ($R_i \in \mathbb{Q} \leftrightarrow$ a rational tangle).

If $R_i = 1/a_i$ for all $i$ ($a_i \in \mathbb{Z}\{0\}$), then we denote it by $P(a_1, \ldots, a_l)$ a pretzel knot of type $(a_1, \ldots, a_l)$. 
Review

For a hyperbolic Montesinos knot with length $l$,

- $l \leq 2 \Rightarrow K$ is 2-bridge knot. All exceptional surgeries are classified [Brittenham-Wu '95].

- $\nexists$ reducible surgery [Wu '96].

- All toroidal surgeries are classified [Wu '06].

- Remaining are SF surgeries with $l = 3$. 

Target: $K = M(R_1, R_2, R_3), K(r) : \text{SF.}$
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\begin{itemize}
  \item $l \leq 2 \Rightarrow K$ is 2-bridge knot. All exceptional surgeries are classified [Brittenham-Wu '95].
  \item $l \geq 4 \Rightarrow \not\exists$ exceptional surgery [Wu '96].
\end{itemize}
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Remains are SF surgeries with $l = 3$.

Target: $K = M(R_1, R_2, R_3), \ K(r) : SF$. 
$r$-surgery on $K$ is cyclic (resp. finite)

$\pi_1(K(r))$ is cyclic (resp. finite).

**Result in the case that** $|\pi_1(K(r))| < \infty$

---

**Theorem 1 [Ichihara-J.]** (arXiv:0807.0905)

$K$ : a hyperbolic Montesinos knot.

(i) If $r$-surgery on $K$ is cyclic,

then $K = P(-2,3,7)$ and $r = 18$ or 19.

(ii) If $r$-surgery on $K$ is acyclic finite,

then $K = P(-2,3,7)$ and $r = 17$, or $K = P(-2,3,9)$ and $r = 22$ or 23.
Together with the result by [Wu], we have:

**Corollary**

Among hyperbolic **arborescent** knots, only the knots in Thm.1 admit such surgeries.
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[Watson]

For \( p \in \{5, 7, \cdots, 25\} \), the \((-2, p, p)\)-pretzel knot admit no finite surgeries.

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[Futer-Ishikawa-Kabaya-Mattman-Shimokawa]

A complete classification of finite surgeries on $(-2, p, q)$-pretzel knots with $p, q$: odd positive.

(arXiv:0809.4278)
Result in the case that $|\pi_1(K(r))| = \infty$

Theorem 2 [Ichihara-J.-Mizushima].

$K$: alternating hyperbolic Montesinos knot.

If $r$-surgery on $K$ is SF, then $K = P(a, b, c)$

with $a = 3$ or 5 and $a < b < c$ : odds.

Key ingredients

- Essential (genuine) lamination
- Symmetries of knots (strong involution & cyclic period)
- The alternation number (The Rasmussen invariant, the signature, #-move, sharper Bennequin inequality, etc.)
- Hyperbolic structure (6-theorem)
§2 Proof of Theorem 1

Theorem 1 [Ichihara-J.] (arXiv:0807.0905)

$K$ : a hyperbolic Montesinos knot.

(i) If $r$-surgery on $K$ is a non-trivial cyclic, then $K = P(-2, 3, 7)$ and $r = 18$ or $19$.

(ii) If $r$-surgery on $K$ is a non-trivial acyclic finite, then $K = P(-2, 3, 7)$ and $r = 17$, or $K = P(-2, 3, 9)$ and $r = 22$ or $23$.

Key ingredients

- Essential lamination
- Heegaard Floer homology
[Outline of Proof of Thm 1.]

\(K\) : a hyperbolic Montesinos knot.

\textbf{Fact 1 [Delman]}

If \(K\) admits a cyclic / finite surgery, then \(K\) is equivalent to either

(i) \(P(-2l, p, q)\),  
(ii) \(P(-1, -1, 2m, p, q)\), or
(iii) \(P(-1, 2n, p, q)\).

Here \(l, m \geq 2, n \neq 0, \text{ and } 3 \leq p \leq q: \text{ odd.}\)


“Constructing essential laminations and taut foliations

which survive all Dehn surgeries”
Every hyperbolic Montesinos knot except for the families (i)–(iii) admits an essential lamination in its exterior which survives after all non-trivial Dehn surgeries (called persistent lamination).

Actually they showed that if a 3-mfd. $M$ contains an essential lamination, then its universal cover must be $\mathbb{R}^3$. In particular, $\pi_1(M)$ is never cyclic/finite.

⇒ We check the families (i)–(iii).
Case 1: $P(-2\ell, p, q)$

**Fact 2 [Mattman]**

If $K$ admits a cyclic / finite surgery, then $K \neq P(-2l, p, q)$ with $l \geq 2$ & $3 \leq p \leq q$: odd.

Mattman,

“Cyclic and finite surgeries on pretzel knots”,

Case 2: $P(-1, -1, 2m, p, q)$

Fact 3 [Ni]

If a knot in $S^3$ admits a cyclic / finite surgery, then it must be a fibered knot.
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**Fact 3 [Ni]**

If a knot in $S^3$ admits a cyclic / finite surgery, then it must be a fibered knot.

**Claim 1**

$P(-1, -1, 2m, p, q)$ is not fibered with $m \geq 2$ and $3 \leq p \leq q$: odd.

Claim 1 is shown by using an algorithm due to [Gabai], which decides fiberedness of a pretzel knot.
Case 3: \( P(-1, 2n, p, q) \)

\[ \text{L-space} \]

A rational homology sphere \( Y \) is an L-space if the rank of \( \widehat{HF}(Y) \) is equal to \( |H_1(Y; \mathbb{Z})| \).

In fact, \( \pi_1(M) : \text{cyclic} / \text{finite} \Rightarrow M : \text{L-space} \).
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**Fact 4 [Ozsváth-Szabó]**

A knot \( K \) in \( S^3 \) admits an integral surgery yielding an L-space \( \Rightarrow \) \( \forall \) non-zero coeff. of \( \Delta_K(t) \) is \( \pm 1 \).
Case 3: $P(-1, 2n, p, q)$

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In fact, $\pi_1(M) : \text{cyclic / finite} \Rightarrow M : \text{L-space}$.

**Fact 4 [Ozsváth-Szabó]**

A knot $K$ in $S^3$ admits an integral surgery yielding an L-space $\Rightarrow \forall$ non-zero coeff. of $\Delta_K(t)$ is $\pm 1$.

**Fact 5**

For a knot $K \subset S^3$, $K(p/q)$ is an L-space $\Rightarrow K(p)$ is also an L-space.
Lemma 2

If $P(-1, 2n, p, q)$ with $n \neq 0$ & $3 \leq p \leq q$: odd admits a cyclic/finite surgery, then $(n, p) = (1, 3)$. 

Proof of Lem. 2

Suppose that $K = P(-1, 2n, p, q)$ admits a cyclic/finite surgery. By Facts 4 & 5, every non-zero coefficient of $\Delta_K(t)$ must be ±1.

Normalization: $\Delta_K(t) = \sum_{i=0}^{\infty} a_i t^i$ with $a_0 > 0$.

Claim 3

• If $n \leq -1$, then $a_1 = 8$ if $n = -1$, $a_1 = 3$ if $n < -2$.

• If $n \geq 2$.

• If $n = 1$ & $3 \leq p \leq q$: odd.
Lemma 2

If $P(-1, 2n, p, q)$ with $n \neq 0$ & $3 \leq p \leq q$: odd
admits a cyclic/finite surgery, then $(n, p) = (1, 3)$.

[Proof of Lem. 2] Suppose that $K = P(-1, 2n, p, q)$
admits a cyclic / finite surgery. By Facts 4 & 5,
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If $P(-1, 2n, p, q)$ with $n \neq 0$ & $3 \leq p \leq q$: odd admits a cyclic/finite surgery, then $(n, p) = (1, 3)$.

[Proof of Lem. 2] Suppose that $K = P(-1, 2n, p, q)$ admits a cyclic / finite surgery. By Facts 4 & 5, every non-zero coefficient of $\Delta_K(t)$ must be $\pm 1$. Normalization: $\Delta_K(t) = \sum_{i=0}^{k} a_i t^i$ with $a_0 > 0$.

Claim 3

• If $n \leq -1$, then $a_1 = \begin{cases} -4 & \text{if } n = -1 \\ -3 & \text{if } n \leq -2 \end{cases}$

• If $n \geq 2$, then $a_3 = 2$.

• If $n = 1$ & $5 \leq p \leq q$: odd, then $a_4 = -2$. 

□ (Lemma 2)
By Lemma 2, if $K$ admits a cyclic/finite surgery, then

$$K = P(-1, 2, 3, q) = P(-2, 3, q)$$

with $q \geq 3$: odd.
By Lemma 2, if $K$ admits a cyclic/finite surgery, then $K = P(-1, 2, 3, q) = P(-2, 3, q)$ with $q \geq 3$: odd.

Then [Mattman] already showed:

Among such knots, only $P(-2, 3, 7) \& P(-2, 3, 9)$ can have cyclic / finite surgeries, and the surgery slopes are the ones in Theorem 1.

\[\square\] (Proof of Theorem 1.)
§3. Outline of the proof of Thm.2

Let $K$ be an alternating hyperbolic Montesinos knot.

**Proposition 1**

If $K$ admits SF surgery, then either

(i) $K = P(a, b, c)$ with $a, b, c \geq 3$: odd,
(ii) $K = P(3, 3, 2n)$ with $n \geq 1$, or
(iii) $K = M\left(\frac{1}{3}, \frac{1}{3}, \frac{2m-1}{2m}\right)$ with $m \geq 2$.

This is proved based on [Delman]:

Proof (Prop. 1)

1) Check whether the Delman’s lami. is genuine,
2) if not, construct “new” one, which is genuine.

□(Prop. 1)
Proposition 2

(1) $K \neq P(a, b, b)$ with $a \geq 2$ and odd $b \geq 3$.

(2) $K \neq M\left(\frac{1}{3}, \frac{1}{3}, \frac{2m-1}{2m}\right)$ with $m \geq 2$.

In the following, we prove Proposition 2.

Key ingredients

- Symmetries of knots (strong involution & cyclic period)
- The alternation number (The Rasmussen invariant, the signature, $#$-move, sharper Bennequin inequality, etc.)

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[Ichihara]

\[ \forall \text{ exceptional surg. on alternating knots is integral.} \]
Outline of Proof of Proposition 2

• The knots $P(a, b, b)$ and $M\left(\frac{1}{3}, \frac{1}{3}, \frac{2m-1}{2m}\right)$ are strongly invertible. Set $K = P(a, b, b)$ or $M\left(\frac{1}{3}, \frac{1}{3}, \frac{2m-1}{2m}\right)$

• Then, by Montesinos trick, $K(r)$ is 2-fold branched covering of a certain link $L_r$.

Remark

$L_r$ is a knot or a 2-comp. link.
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- Then, by Montesinos trick, $K(r)$ is 2-fold branched covering of a certain link $L_r$.
- If $K(r)$ is SF, then $L_r$ is either a Montesinos link or a Seifert link (Seifert link = the exterior is SF).

Remark

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Outline of Proof of Proposition 2

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• Then, by Montesinos trick, $K(r)$ is 2-fold branched covering of a certain link $L_r$.

• If $K(r)$ is SF, then $L_r$ is either a Montesinos link or a Seifert link (Seifert link = the exterior is SF).

• We actually show that $L_r$ is neither a Montesinos link nor a Seifert link. Then we complete the proof of Proposition 2.

Remark

$L_r$ is a knot or a 2-comp. link.
• By using the $\det(L)$, $g(L)$, and $\Delta_L(t)$, we can show that $L_r$ is not a Seifert link.
• We can easily show that $L_r$ is not a 2-comp. Montesinos link.
• By using the $\det(L)$, $g(L)$, and $\Delta_L(t)$, we can show that $L_r$ is not a Seifert link.
• We can easily show that $L_r$ is not a 2-comp. Montesinos link.
• To show that $L$ is not a Montesinos knot, we use the alternation number.

We will explain the last claim.
Here we introduce the alternation number.

**alternation number [Kawauchi]**

\[ \mathcal{A} \equiv \{ \text{alternating links} \} (\ni \text{trivial links}). \]

\[ d_G(\cdot, \cdot) : \text{the Gordian distance}. \]

\[ \text{alt}(L) = \min_{L' \in \mathcal{A}} d_G(L, L'). \]

**Remark**

- \( \text{alt}(K) \leq u(K) \)
- \( \forall n \in \mathbb{N}, \exists K \text{ s.t. } \text{alt}(K) = n. \) [Abe], [Kawauchi]
By using the following two facts, we can show that the link $L_r$ is not a Montesinos knot.

**Fact [Abe-J.-Kishimoto]**

\[
\text{alt}(L) \leq 1 \quad \text{for any Montesinos link } L.
\]

**Fact [Abe]**

For a knot $K$, \( \text{alt}(K) \geq \frac{|s(K) - \sigma(K)|}{2} \)

where \( s(K) \): the Rasmussen invariant of $K$, and \( \sigma(K) \): the signature of $K$

with \( \sigma(\text{right-handed trefoil}) = 2 \).

Actually, we can show that \( s(L_r) - \sigma(L_r) \geq 4 \).