Bounds on numerical boundary slopes for Montesinos knots

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Preprint:
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“Bounds on numerical boundary slopes for Montesinos knots”
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§1. Definitions and Notations

\( M \) : a compact orientable 3-manifold with single toral boundary \( \partial M \).

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**Definition (essential surface)**

A compact, connected surface \( F \) properly embedded in \( M \) is called **essential** if it is incompressible and not boundary parallel.
Definition (slope)
A slope on $\partial M$ is the isotopy class of a simple closed curve on $\partial M$.

Definition (boundary slope)
The $\partial$-slope of $F$ is the slope determined by boundary components of $F$. 
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A slope on $\partial M$ is the isotopy class of a simple closed curve on $\partial M$.

**Definition (boundary slope)**
The $\partial$-slope of $F$ is the slope determined by boundary components of $F$.

**Facts**
$\partial$-slopes are only finitely many (Hatcher).
There are at least two $\partial$-slopes (Culler-Shalen).
If a meridian-longitude system on $\partial M$ is fixed, a slope is represented by a rational number or $\infty$. i.e. \[
\{ \text{slopes on } \partial M \} \leftrightarrow \mathbb{Q} \cup \{\infty\}\]
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\{\text{slopes on } \partial M\} \leftrightarrow \mathbb{Q} \cup \{\infty\}
\]

**Definition**
A slope regarded as a rational number is called a numerical slope.

**Goal**
Study numerical properties of $\partial$-slopes for Montesinos knots.
Montesinos knot $K(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_n}{q_n})$

Assume that:

- the number of tangles $n$ is at least 3,
- all fractions are non-integral.
§2. Results

In the following, we assume that all surfaces are orientable just for simplifying the statements.

In fact, we have general results (including non-orientable case).

Please see our preprint.
2-1. Upper bound on difference

Let $r_1, r_2$ be $\partial$-slopes of essential surfaces of genus $g_1, g_2$ for a Montesinos knot.

Consider the difference $|r_1 - r_2|$ of two $\partial$-slopes.
2-1. Upper bound on difference

Let $r_1, r_2$ be $\partial$-slopes of essential surfaces of genus $g_1, g_2$ for a Montesinos knot.

Consider the difference $|r_1 - r_2|$ of two $\partial$-slopes.

**Theorem 1.**

$$|r_1 - r_2| \leq 4 (g_1 + g_2)$$
c.f.

Theorem. [I]

In general, if $K$ is hyperbolic, then $|r_1 - r_2| \leq 12 (g_1 + g_2 - 1)$
Theorem. [I]
In general, if \( K \) is hyperbolic, then \( |r_1 - r_2| \leq 12 (g_1 + g_2 - 1) \)

Corollary 1. (Toroidal Surgery)
If \( r \)-surgery on a Montesinos knot \( K \) yields a \text{toroidal} 3-mfd, then \( |r| \leq 4g + 4 \), where \( g \) denotes the genus of \( K \).
Definition (Distance between slopes)
The distance $\Delta(r_1, r_2)$ of two slopes means the minimal geometric intersection number of their representatives.

Remark that

(1) $\Delta$ is not a distance in the usual sense.

(2) For $r_i = p_i/q_i$, $\Delta(r_1, r_2) = |p_1 \cdot q_2 - p_2 \cdot q_1|$. 
Corollary 2. (bound on $\Delta$)

For a non-torus Montesinos knot,

$\Delta(r_1, r_2) < 8(2g_1 - 1)(2g_2 - 1)$
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For a non-torus Montesinos knot,
$$\Delta(r_1, r_2) < 8(2g_1 - 1)(2g_2 - 1)$$

Facts. (Torisu)

For a non-cabled knot,
$$\Delta(r_1, r_2) < 36(2g_1 - 1)(2g_2 - 1)$$

Facts. (Hass-Rubinstein-Wang)

For a hyperbolic knot,
$$\Delta(r_1, r_2) < \frac{43}{4}(2g_1 - 1)(2g_2 - 1)$$
2-2. Upper bound on denominator

Let \( r = \frac{p}{q} \) be a \( \partial \)-slope of an essential surface of genus \( g \) for a Montesinos knot (\( q \geq 1 \)).

Consider the denominator \( q \) of the \( \partial \)-slope.
2-2. Upper bound on denominator

Let \( r = p/q \) be a \( \partial \)-slope of an essential surface of genus \( g \) for a Montesinos knot \((q \geq 1)\).

Consider the denominator \( q \) of the \( \partial \)-slope.

**Theorem 2.**

\[
\begin{cases}
q \leq g + 1 & (g = 0, 1) \\
q \leq 2g - 1 & (g \geq 2).
\end{cases}
\]

Remark: this is best possible.
Facts.

(Torisu) For a composite knot, $q \leq g$.

(Menasco-Thistlethwaite) For an alternating knot, $q \leq g$.

(Gordon-Luecke) If $g = 0$, then $q = 1$. 
Facts.

(Torisu) For a composite knot, $q \leq g$.

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For an alternating knot, $q \leq g$.

(Gordon-Luecke) If $g = 0$, then $q = 1$.

Corollary 3. (c.f. Eudave-Munoz)

No Montesinos knot other than torus knot admits an essential surface of genus 0. Thus the Cabling Conjecture is true for Montesinos knots.
§3. Key of Proofs

**Algorithm by Hatcher and Oertel**

: finds and enumerates all boundary slopes for a given Montesinos knot.

: is based on Algorithm by Hatcher and Thurston for two-bridge knots.

: seems to be algebraic or combinatorial.

(i) produces candidate surfaces.

(ii) verifies incompressibility of surfaces.
Dunfield’s software implements Algorithm of Hatcher and Oertel.

is available from

http://www.its.caltech.edu/~dunfield/montesinos/index.html

is written in Python language.

Computer experiments have been very helpful!!
In algorithm,

essential surface

\[ \updownarrow \]

system of sequences of irreducible fractions

(we call an edge path system)

Example. A Seifert surface for \( K(-\frac{1}{2}, \frac{1}{3}, \frac{1}{7}) \)

\[
\begin{align*}
\langle \infty \rangle & - \langle \frac{0}{1} \rangle - \langle -\frac{1}{2} \rangle \\
\langle \infty \rangle & - \langle \frac{1}{1} \rangle - \langle \frac{1}{2} \rangle - \langle \frac{1}{3} \rangle \\
\langle \infty \rangle & - \langle 1 \rangle - \langle \frac{1}{2} \rangle - \langle \frac{1}{3} \rangle - \langle \frac{1}{4} \rangle - \langle \frac{1}{5} \rangle - \langle \frac{1}{6} \rangle - \langle \frac{1}{7} \rangle
\end{align*}
\]
Decompose $S^3$ into balls $B_1, \cdots, B_n$. $B_i \supset T_i$
An essential surface $F$ can be isotoped to be in a **standard position** in each $B_i$:

- (a) base disks
- (b) saddle
- (c) compound
- (d) cap
In $B_i \supset S^2 \times [0, 1]$, the intersection curves of $F$ and level spheres give an edge path system.
We can use many numerical properties.

For example: \[ \langle \frac{p}{q} \rangle - \langle \frac{r}{s} \rangle \iff |p \cdot s - q \cdot r| = 1 \]
Our next project is to get a lower bound on diameter of the set of \( \partial \)-slopes.

Let \( r_1, r_2 \) be the greatest and least \( \partial \)-slopes for a knot \( K \).

**Definition.** The difference \( |r_1 - r_2| \) of the two \( \partial \)-slopes is called the **diameter** of the set of \( \partial \)-slopes, and denoted by \( Diam(K) \).
Facts.

(Culler-Shalen) \( Diam(K) \geq 2 \)

(Ishikawa-Mattman-Shimokawa)

\[ Diam(K) > \frac{\|\beta\|}{q\|\mu\|} \]
Facts.

(Culler-Shalen) \( Diam(K) \geq 2 \)

(Ishikawa-Mattman-Shimokawa)

\[ Diam(K) > \frac{||\beta||}{q||\mu||} \]

Problem.

What about for Montesinos knots?
Are there sharper bounds
(related to genera of surfaces) ?

We performed experiments about relation between \( Diam(K) \) and topological quantities of surfaces.
Computer Experiments:

Via Dunfield’s computer program, we performed the following process iteratively:

- Find the greatest and least $\partial$-slopes $r_1, r_2$.
- Find essential surfaces $F_i$ with $\partial$-slope $r_i$.
- Calculate $r_1 - r_2$ and $(-\chi_1/\#s_1) + (-\chi_2/\#s_2)$.

By plotting the obtained data, we have graphs.
(a) Montesinos knot: \[ K = K\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right) \]

Conditions:

- \(|p_i| \leq 7, \ 2 \leq q_i \leq 7\)
- \(p_i/q_i\) is not an integer.
- \(p_i/q_i\) is an irreducible fraction.
- \(K\) must be a knot (not a link).

<table>
<thead>
<tr>
<th>Set of three fractions</th>
<th>175,616</th>
</tr>
</thead>
<tbody>
<tr>
<td>The number of Knots</td>
<td>95,952</td>
</tr>
</tbody>
</table>
Montesinos knots \((n = 3)\)

\[
Diam = \left( -\frac{\chi_1}{\#s_1} \right) + \left( -\frac{\chi_2}{\#s_2} \right)
\]
(b) Pretzel knot \( K = P(q_1, q_2, q_3) \)

\[ = \text{Montesinos knot} \quad K(\frac{\pm 1}{q_1}, \frac{\pm 1}{q_2}, \frac{\pm 1}{q_3}) \]

Conditions:

- \( 2 \leq q_i \leq 20 \)
- \( K \) must be a knot (not a link).

<table>
<thead>
<tr>
<th>Set of three fractions</th>
<th>54,872</th>
</tr>
</thead>
<tbody>
<tr>
<td>The number of Knots</td>
<td>25,272</td>
</tr>
</tbody>
</table>
\[ (-\chi_1/#s_1) + (-\chi_2/#s_2) \]
Consequently, we have:

Conjecture

\[ Diam(K) \geq 2\left(\frac{-\chi_1}{\#s_1} + \frac{-\chi_2}{\#s_2}\right) \]

In particular, \( Diam(K) \geq 4 \).