Hyperbolic structures
and Dehn surgeries
on the figure-eight knot
complement

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§1 How to decompose $S^3 - \bigcirc = \bigtriangleup \cap \bigtriangleup$

§2 Hyperbolic structures on $S^3 - \bigcirc$

§3 Dehn surgery and the $2\pi$-theorem

**Goal** To show

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**Theorem**

On the figure-eight knot, all but at most 12 Dehn surgeries yield 3-manifolds with a metric of **negative curvature**.

by using the Gromov-Thurston's $2\pi$-theorem.
§1 How to decompose $S^3$—

Span 4 surfs!
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Span 4 surf!
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Consider the cylinder $X$.

Cut $S^3 - K$ along $A \sim D$ into $T_1, T_2$. 
Consider the cylinder $X$.

Cut $S^3 - K$ along $A \sim D$ into $\tau_1, \tau_2$. 

$S^3$
Cut $S^3 - K$ along $A \sim D$ into $\tau_1, \tau_2$.

Consider the cylinder $X$.
Consider the cylinder $X$.

Cut $S^3 - K$ along $A \sim D$ into $\tau_1, \tau_2$. 
Consider the side of $\tau_1$ only.

Cutting along $e_1$.
\[ S^3 - \text{Link} = \tau_1 \cup \tau_2 \]
§2 Hyperbolic structures on $S^3$.

Let $M$ be a knot complement having a decomposition by ideal tetrahedra.

**Question.**

When does $M$ admit a hyperbolic structure?

**Ans.**

It is the case that the glueing conditions are satisfied.
**Poincaré model**

The unit ball $B^3$, i.e.,

$$\{ (x, y, z) \in \mathbb{R}^3 \mid r := \sqrt{x^2 + y^2 + z^2} < 1 \}$$

with the metric

$$\frac{dx^2 + dy^2 + dz^2}{(1 - r^2)^2}.$$

**Upper half space model**

$\mathbb{R}_+^3 := \{ (x, y, t) \in \mathbb{R}^3 \mid t > 0 \}$

with the metric $\frac{dx^2 + dy^2 + dt^2}{t^2}$. 
To describe the conditions, we introduce the modulus of an edge \( e \) in an ideal tetrahedron \( \tau \).

Let \( Z_\tau(e) \) be the complex number corresponding to \( v_2 \), and call it the modulus of the edge \( e \).
The glueing conditions
Around each edge, if
1. the product of the moduli is 1 and
2. the sum of the dihedral angles is $2\pi$,
then $M$ admits a hyperbolic structure.

\[\text{not satisfied.}\]

\[\text{View from } \infty.\]

\section*{Remark}
The structure may be incomplete.
For the figure-eight knot complement,

\[ S^3 - K = \tau_1 \cup \tau_2, \]

if \( \tau_1, \tau_2 \) are ideal regular tetrahedra
(i.e., all dihedral angles are \( \pi/3 \)),

then the glueing conditions are satisfied,
and \( S^3 - K \) has a complete hyperbolic structure. (For completeness, details will be given in the next talk.)
§3 Dehn surgery and the $2\pi$-theorem

Let $K$ be a knot in a 3-manifold $M$ and $N(K)$ denote the regular neighborhood of $K$ in $M$.

\[ M \mapsto N := \bigg( M - \overset{\circ}{N}(K) \bigg) \bigcup_{f} \text{(solid torus } V \bigg), \]

where $f : \partial V \to \partial N(K)$ is a homeo.

We say that $N$ is obtained by a Dehn surgery on $K$.

**Remark.** A Dehn surgery is determined by the isotopy class of the curve $f(\text{meridian of } V)$ on $\partial N(K)$. 
Hyperbolic Dehn Surgery Theorem
All but finitely many Dehn surgeries on a hyperbolic knot (i.e., a knot with the complete hyperbolic complement) yield closed hyperbolic 3-manifolds.

In the rest of the talk, we will show

Theorem
On the figure-eight knot, all but at most 12 Dehn surgeries yield 3-manifolds with a metric of negative curvature.

To show this, we use the Gromov-Thurston’s $2\pi$-theorem.
Let $K$ be a hyperbolic knot in a 3-manifold $M$ and take $N(K)$ s.t. $\partial N(K)$ is a horotorus.

Then $\partial N(K)$ has an Euclidean structure.

Let $N$ be the manifold obtained by Dehn surgery on $K$, i.e.,

$$N = \left( M - \overset{\circ}{N}(K) \right) \cup_{f} (\text{solid torus } V),$$

where $f : \partial V \to \partial N(K)$ is a homeo.

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The Gromov-Thurston's $2\pi$-theorem

If the length of $f(\text{meridian of } V) > 2\pi$, then $N$ has a metric of negative curvature.
Outline of the proof.

We will construct a negatively curved metric on $V$. On $V \cong D^2 \times S^1$, 

\[ ds^2 := dr^2 + (f(r))^2 d\mu^2 + (g(r))^2 d\lambda^2, \]

where $f(r_0) = 0$.

We will describe the conditions that 

(1) $ds^2$ is a non-singular metric, 

(2) $ds^2$ has negative sectional curvatures, 

(3) $ds^2$ mutchs with the metric on $E(K)$, 

where $E(K)$ denotes $M - \overset{\circ}{N}(K)$. 
(1) The singularity appears along the core circle of $V$. The cone-angle is

$$\lim_{r \to r_0} \frac{1}{r - r_0} \int_0^1 f(r) d\mu = f'(r_0).$$

Hence, the condition is $f'(r_0) = 2\pi$.

(2) By direct calculations from Riemannian geometry, the condition is

$$\frac{f''}{f}, \frac{g''}{g}, \frac{f'g'}{fg} > 0.$$
Near \( \partial E(K) \), the metric on \( E(K) \) is of the above type with

\[
f(r) = \ell_1 e^r, \quad g(r) = \ell_2 e^r,
\]

where \( \ell_1, \ell_2 \) are some constants.

A face of tetrahedra.
Here, note that $l_1$ equals to the length of $f(\text{meridian of } V)$.

$$l_1 = \int_0^1 a_i e^r \, dr,$$  \hspace{1cm} \text{at } r = 0

\text{If } l_1 > 2\pi,  \\
\text{conditions (1), (2), (3) are satisfied !!}
Now, let $K$ be the figure-eight knot.

**Question**

How many closed (geodesic) curves on $\partial N(K)$ of length $\leq 2\pi$?

Take the maximal $N(K)$.

![Diagram](just before touch!)

Then, $\partial N(K)$ has the following Euclidean structure.
Consider the universal cover of $\partial N(K)$.

There are 12 curves!!

In the next talk, we will see that all but at most 10 Dehn surgeries yield hyperbolic 3-manifolds.