Two-bridge knots admit no purely cosmetic surgeries

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joint work with
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Dehn surgery on a knot

$K$ : a knot in a 3-manifold $M$

**Dehn surgery on $K$**

1) remove the open tubular neighborhood of $K$ from $M$
2) glue a solid torus back (along a slope $\gamma$)
Cosmetic surgery conjecture

It is natural to ask:

Can distinct Dehn surgeries give the same manifold?

Conjecture. [Problem 1.81(A) in Kirby’s list]

Two Dehn surgeries on inequivalent slopes are never purely cosmetic.

- Two slopes for a knot $K$ are called *equivalent* if $\exists$ homeo. of the exterior of $K$ taking one slope to the other.
- Two surgeries on $K$ are called *purely cosmetic* if $\exists$ orientation preserving homeo. between the manifolds obtained by the surgeries.
Main result

Theorem [Ichihara-J.-Mattman-Saito]
Two-bridge knots admit no purely cosmetic surgeries.

Our argument, based on a recent result by Hanselman, uses several invariants of knots or 3–manifolds; for knots, the signature and some finite type invariants, and for 3–manifolds, the $SL(2, \mathbb{C})$ Casson invariant.

Also, we have the following.

Theorem [Ichihara-J.-Mattman-Saito]
All alternating fibered knots and all alternating pretzel knots admit no purely cosmetic surgeries.
Hanselman’s result

\( K \): an alternating knot in \( S^3 \)
\( g(K) \): the genus of \( K \)
\( \sigma(K) \): the signature of \( K \)

**Lemma ([Hanselman, arXiv:1906.06773])**

If \( K \) admits purely cosmetic surgeries, then \( g(K) = 2, \sigma(K) = 0 \), and the surgery slopes must be either \( \pm 1 \) or \( \pm 2 \).

The latter two assertions follow from [Hanselman, Theorem 5] directly. Also from the same theorem, the Alexander polynomial of \( K \) must be \( \Delta_K(t) = nt^2 - 4nt + (6n + 1) - 4nt^{-1} + nt^{-2} \) for some positive integer \( n \). Then, by the work of Murasugi and Crowell, the genus \( g(K) \) of \( K \) must be 2.
By using Jones polynomial of knots, we have the following.

**Lemma ([Ichihara-Wu, 2019])**

If a 2-bridge knot of genus two admitted purely cosmetic surgeries, then it would be associated to the continued fraction $[2x, 2y, -2(x+y), 2x]$ for integers $x > 0$ and $y \neq 0$.

Sample picture:

![Diagram of knots]

Figure 1: Diagram of knots $[2x, 2y - 1, 1, 2x + 2y - 2, 1, 2x - 1]$
Signature $\sigma(K)$

Let $K$ be a two-bridge knot associated to the continued fraction $[2x, 2y, -2(x + y), 2x]$ for integers $x > 0$ and $y \neq 0$.

**Proposition 1.**

If $K$ admits purely cosmetic surgeries, then $y < 0$ and $(x + y) > 0$.

We use the following result of [Lee] and [Traczyk]:
for a reduced alternating diagram $D$ of an oriented non-split alternating link $L$,

$$\sigma(L) = o(D) - y(D) - 1$$

It remains to handle the case of $y < 0$ and $(x + y) > 0$. In this case, the simple continued fraction for $K$ is $[2x - 1, 1, -(2y + 1), 2(x + y) - 1, 1, 2x - 1]$. 
Let $K$ be a two-bridge knot associated to the continued fraction $[2x - 1, 1, -(2y + 1), 2(x + y) - 1, 1, 2x - 1]$ for some $x > 0$, $y < 0$ with $(x + y) > 0$.

**Proposition 2.**

If $K$ admits purely cosmetic surgeries, then $x = -2y$.

Note that the knot is amphichiral when $x = -2y$.

Our key ingredient is the $SL(2, \mathbb{C})$ Casson invariant, originally introduced by [Curtis].

A practical surgery formula for two-bridge knots was obtained by [Boden-Curtis], and was used for a study of cosmetic surgeries on two-bridge knots in [Ichihara-Saito].
Finite type invariants

Note that if $x = -2y$, then the knot $K$ is associated to the continued fraction $[4n, -2n, -2n, 4n]$ for $n > 0$.

**Proposition 3.**

The two-bridge knot $K$ associated to the continued fraction $[4n, -2n, -2n, 4n]$ for a positive integer $n$ admits no purely cosmetic surgeries.

We use the obstructions obtained by [Boyer-Lines] and [Ito]: If a knot $K$ has a purely cosmetic pair of surgeries, then

- $a_2(K) = 0$
- $j_4(K) \neq 14n^4$ and $j_4(K) \neq 284n^4$ for some $n > 0$.

On the other hand, by direct calculations, we have

$$\nabla_K(z) = 1 + 4n^4z^4 \quad \text{and} \quad j_4(K) = -12n^4$$

for $K = C[4n, -2n, -2n, 4n]$ with $n > 0$. 