Integral non-hyperbolike surgeries

Kazuhiro Ichihara

Osaka Sangyo Univ.
Notation

\begin{align*}
S^3 \supset K & : \text{knot} \\
N(K) & : \text{regular nhd of } K \\
E(K) & : \text{exterior of } K \\
& \quad \text{(i.e., } S^3 - \overset{\circ}{N}(K))
\end{align*}
Dehn surgery

\[ S^3 \]

\[ \rightarrow \]

Solid torus

\[ E(K) \cup (\text{solid torus}) \]

**Hyperbolic Surgery Thm.**

Only finitely many surgeries on a hyperbolic knot yields non-hyperbolic manifolds.

Surgeries yielding non-hyperbolic 3-mfds are called exceptional.
Problem

How many exceptional surgeries occur?

Recall that:

Definition

A 3-mfd is called hyperbolic if it admits a metric of constant curvature $-1$.

A knot $K$ is called hyperbolic if $S^3 - K$ is hyperbolic.

Fact $K$ is hyperbolic if and only if $K$ is neither a satellite knot nor a torus knot.
Conjecture (Gordon)

Any hyperbolic knot admits at most 10 exceptional surgeries. Moreover, if there are 10, then $K$ is the figure-eight knot.

Theorem (Hodgson–Kerckhoff)

Any hyperbolic knot admits at most 60 exceptional surgeries.

Theorem (Agol, Lackenby)

Any hyperbolic knot admits at most 12 surgeries yielding non-hyperbolike 3-mfds.
Definition

A 3-mfd $M$ is Hyperbolike if $\pi_1(M)$ is non-elementary word-hyperbolic.

Remark: $M$ is Hyperbolic

$\Downarrow$

$M$ is Hyperbolike

$\Downarrow$

$M$ is neither Reducible, Toroidal, nor Small Seifert fibered.

If Geometrization Conjecture is TRUE, these are all equivalent.
Theorem

Any hyperbolic knot admits at most 10 integral surgeries yielding non-hyperbolike 3-mfds.

Corollary

‘Almost all’ arborescent knots admit at most 10 surgeries yielding non-hyperbolike 3-mfd.
Recall that:

\[
\{ \text{Dehn surgery on } K \} \\
\implies \text{parametrized} \\
\begin{cases}
\text{isotopy class of loops on } \partial N(K) \text{ (called slope)}
\end{cases}
\]

\[
\{ \text{slope} \} \cong \{ p[m] + q[\ell] \}/\pm \\
\cong \{ p/q \} \cup \{ 1/0 \} \cong \mathbb{Q} \cup \{ 1/0 \}
\]

where

\[
m : \text{meridian} \quad (\text{i.e., } [m] = 0 \text{ in } H_1(N(K)))
\]

\[
\ell : \text{longitude} \quad (\text{i.e., } [\ell] = 0 \text{ in } H_1(E(K)))
\]
Outline of Proof

Regard $S^3 - K$ as a cusped hyperbolic 3-mfd.

Take the maximal horotorus $T$ ($T$ is topologically $\partial N(K)$).

Note

$T$ can be regarded as a Euclidean torus.

Define the length of slope $r$ as the minimal length of the loop on $T$ with slope $r$. 
Use the following facts:

**Fact 1 (Agol, Lackenby)**
If surgery along $r$ yields a non-hyperbolilke 3-mfd, then $\text{length}(r) \leq 6$.

**Fact 2 (Adams)**
If $K$ is not the knot $4_1$, $5_2$, then $\text{length}(m) > \frac{4}{\sqrt{2}}$.

**Fact 3 (Cao-Meyerhoff)**
The Area of $T$ is at least 3.35.