On exceptional surgeries on Montesinos knots

K. Ichihara

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On alternating Montesinos knots

Joint work with

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(OCAMI)

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東大トポロジー火曜セミナー，June 15, 2010
1. Introduction

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Aims of Today’s Talk

To give an introduction;

a study on Dehn surgery by using knot invariants

In particular, we focus on

- \( HF(K) \) : the Heegaard Floer homology
- \( s(K) \) : the Rasmussen invariant
  (derived from khovanov homology)
This talk is based on

- K. Ichihara and I.D. Jong
  Cyclic and finite surgeries on Montesinos knots

- K. Ichihara, I.D. Jong and S. Mizushima
  Seifert fibered surgeries on alternating Montesinos knots
  in preparation.
3-manifolds

3-dimensional manifold (3-manifold)

A topological space, which locally looks like the 3-dimensional Euclidean space.

Example: Our Universe
Classification of 3-manifolds

Every closed orientable 3-manifold is;

- **Reducible** (containing essential 2-sphere),
- **Toroidal** (containing essential torus),
- **Seifert fibered** (foliated by circles), or
- **Hyperbolic** ($\exists$ Riem. metric of curv. $-1$).

as a consequence of the Geometrization Conjecture
including famous Poincaré Conjecture (1904)
conjectured by Thurston (late '70s)
established by Perelman (2002-03)
What’s the NEXT?

• Attack the remaining **Open Problems**.
  (e.g., Virtually Haken Conjecture . . . )

• Relate **Geometric & Topological** invariants
  (e.g., Volume conjecture . . . )

• Study the **Relationships** between 3-mfds.
  (e.g., **Dehn surgery** . . . )
  (⇑ **Today**)
**Dehn surgery on knot**

\( E(K) \): the exterior of a knot \( K \) in a 3-mfd \( M \)

(i.e., \( M - \) (open tubular nbd of \( K \))

Gluing a solid torus \( V \) to \( E(K) \) along slope \( \gamma \)

**Thm. [Wallace ('60), Lickorish ('62)]**

Every pair of closed orientable 3-manifolds are related by a finite sequence of Dehn surgeries.
Surgery slope

Dehn surgery on a knot $K$ is determined by the slope $\gamma$ (i.e., isotopy class of simple loop) on the peripheral torus $T$ of $K$;

$$\gamma = \left[ f(\text{meridian of } V) \right]$$

$K(\gamma)$: the manifold obtained by surgery on $K$ along the slope $\gamma$.
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**Parametrization of slope**

When the knot is in the 3-sphere $S^3$, by a standard meridian-longitude system, slopes correspond to irreducible fractions.

\[
\{\text{slope on } T}\ \overset{1:1}{\longleftrightarrow}\ \left\{ \frac{p}{q} \right\} \cup \left\{ \frac{1}{0} \right\}
\]

**Example**

- $1/0$ represents the **meridian** (i.e., bounding a disk in the nhd. of the knot).
- $0$ represents the **preferred longitude** (i.e., bounding a surface in the knot exterior).
Dehn surgeries on a hyperbolic knot giving non-hyperbolic mfds

[Thurston] They are only finitely many.

Each exceptional surgery is either:

- **Reducible** (essential 2-sphere)
- **Toroidal** (essential torus)
- **Seifert fibered** (foliation by circles)

Remark: a hyperbolic knot $= M - K$ is hyperbolic
Montesinos knot

arcs on a 4-punctured sphere, and $\frac{1}{2}$-tangle

**length** of the knot

$\equiv$ minimal number of rational tangles

In particular, a Montesinos knot $K$ is called

a $(a_1, \cdots, a_n)$-pretzel knot, $P(a_1, \cdots, a_n)$

if the rational tangles are $\frac{1}{a_1}, \cdots, \frac{1}{a_n}$. 
For a hyperbolic Montesinos knot with length $l$,

- $l \leq 2 \Rightarrow K$ is 2-bridge knot. All exceptional surgeries are classified [Brittenham-Wu '95].
- $l \geq 4 \Rightarrow \exists$ exceptional surgery [Wu '96].
- $\exists$ reducible surgery [Wu '96].
- All toroidal surgeries are classified [Wu '06].

$\Rightarrow$ Target: $K = M(R_1, R_2, R_3) \& K(r) : SF.$
Remark:

[non-hyperbolic Montesinos knots]

Montesinos knot of length $\leq 2 \Rightarrow$ two-bridge. Then, by [Menasco]
non-hyperbolic two-bridge knots are $T_{(2,p)}$.

Otherwise, non-hyperbolic Montesinos knots:
$P(-2,3,3) \& P(-2,3,5)$, $(= T_{(3,4)} \& T_{(3,5)})$. (by [Oertel], [Bonahon-Siebenmann])

Dehn surgeries on torus knots $T_{(p,q)}$
have been completely classified by [Moser].
2. Cyclic/Finite surgery
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Problem

On (hyperbolic) knots in $S^3$, determine all Dehn surgeries giving 3-mfds with cyclic or finite fundamental groups.

We call such surgeries cyclic surgeries / finite surgeries respectively.

Remark: such mfds are all Seifert fibered.
Such surgeries would be very “special”
⇒ they would be severely restricted.

- On non-hyperbolic knots,
such surgeries have been classified.

- On each hyperbolic knots;
  Cyclic/Finite surgeries are at most Three/Five
  [Culler-Gordon-Luecke-Shalen]/[Boyer-Zhang]
We give a complete classification of cyclic / finite surgeries on Montesinos knots.

**Theorem 1**

If a hyperbolic Montesinos knot $K$ admits;

(i) a non-trivial cyclic surgery along $\gamma$,

then $K \cong P(-2, 3, 7)$ and $\gamma = 18$ or 19,

(ii) a non-trivial acyclic finite surgery $\gamma$,

then $K \cong P(-2, 3, 7)$ and $\gamma = 17$, or $K \cong P(-2, 3, 9)$ and $\gamma = 22$ or 23.
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Related results

[Watson]: for $p \in \{5, 7, \cdots, 25\}$,
Surgery obstructions from Khovanov homology.
(by using Khovanov homology)

[Futer-Ishikawa-Kabaya-Mattman-Shimokawa]:
a complete classification of finite surgeries on
$(-2, p, q)$-pretzel knots with $p, q$: odd positive.
Outline of Proof of Thm 1.

\( K \): a hyperbolic Montesinos knot

**Fact 1. [Delman]**

If \( K \) admits a cyclic / finite surgery, then \( K \) is equivalent to either
\[ P(-2\ell, p, q), P(-1, 2n, p, q), \text{ or } P(-1, -1, 2m, p, q) \]
with \( \ell > 1, n \neq 0, m > 1 \& 3 \leq p \leq q \): odd.


“Constructing essential laminations and taut foliations which survive all Dehn surgeries”
Every hyperbolic Montesinos knot except for the three families admits an essential lamination in its exterior surviving after all non-trivial Dehn surgeries.

Essential lamination (introduced by [Gabai-Oertel])

They showed that if a 3-mfd. $M$ contains an essential lamination, then its universal cover must be $\mathbb{R}^3$.

In particular, $\pi_1(M)$ is never cyclic/finite.
Case 1: \( P(-2\ell, p, q) \)

Fact 2. [Mattman]

If \( K \) admits a cyclic / finite surgery, then
\( K \not\cong P(-2\ell, p, q) \) with \( \ell > 1 \) & \( 3 \leq p \leq q \): odd.

T.W. Mattman,

“Cyclic and finite surgeries on pretzel knots”,

KEY: Culler-Shalen norm (introduced and studied mainly by [Culler-Shalen] and [Boyer-Zhang]); essentially defined by using the character variety of \( SL(2, \mathbb{C}) \)- or \( PSL(2, \mathbb{C}) \)-representations.
For the remaining cases

Use “Heegaard Floer homology”

Case 2: $P(-1, 2n, p, q)$

Fact 3. [Ozsváth-Szabó]

if a knot $K$ in $S^3$ admits an integral Dehn surgery yielding an L-space, then every non-zero coeff. of the Alexander polynomial $\Delta_K(t)$ is $\pm 1$. 
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P. Ozsváth and Z. Szabó,
On knot Floer homology and lens space surgeries,
Topology 44 (2005), 1281–1300.

Here a rational homology sphere $Y$ is an L-space if the rank of $\widehat{HF}(Y)$ is equal to $|H_1(Y; \mathbb{Z})|$.

In fact,

$M$ has $\pi_1(M)$; cyclic / finite $\Rightarrow$ $M$ is an L-space which is shown by [Ozsvath-Szabo].
Alexander polynomial $\Delta_K(t)$

One of the “oldest” invariant in Knot Theory.

- J.W. Alexander defined in 1928 homologically
- Also he found it derived from the knot group
- combinatorial recursive formula (Conway, 1970)

Also related to;

deformation of representations of knot groups
representation of braid groups (Burau representation)
Reidemeister torsion (Milnor)
Casson invariant $\alpha_2$
Seiverg-Witten invariant,
and, Heegaard Floer homology (categorification)
Case 2. (continued)

Suppose;
\[ P(-1, 2n, p, q) \] with \( n \neq 0 \) & \( 3 \leq p \leq q \): odd admits a cyclic/finite surgery.

Claim 1.
Let \( K \) be a knot in \( S^3 \).
If \( p/q \)-surgery on \( K \) yields an L-space, then \( p \)-surgery on \( K \) also yields an L-space.

By Claim 1. & Fact 3.,
every non-zero coefficient of \( \Delta_K(t) \) must be \( \pm 1 \).
Calculations of $\Delta_K(t)$

Notation & Normalization

- The $j$-th coeff. of $\Delta_K(t)$ is denoted by $[\Delta_K(t)]_j$.

Claim 2.

- If $n \leq -1$, then 
  
  $$[\Delta_K(t)]_1 = \begin{cases} 
  -4 & \text{if } n = -1 \\
  -3 & \text{if } n \leq -2
  \end{cases}$$

- If $n \geq 2$, then $[\Delta_K(t)]_3 = 2$.

- If $n = 1$ & $5 \leq p \leq q$: odd, then 
  
  $$[\Delta_K(t)]_4 = -2.$$
By Claim 2 ...

⇒ \((n, p) = (1, 3)\).

⇒ \(K = P(-1, 2, 3, q) \cong P(-2, 3, q)\) with \(q \geq 3\text{:odd} \).

Then \[\text{Mattman}\] already showed:

Among such knots, only \(P(-2, 3, 7) \& P(-2, 3, 9)\) can have cyclic / finite surgeries, and the surgery slopes are the ones in Thm. 1.
Case 3: $P(-1, -1, 2m, p, q)$

Fact 4. [Ni]

If a knot in $S^3$ admits a surgery yielding L-space, then it must be a fibered knot.

Y. Ni,

Knot Floer homology detects fibred knots,


A knot is called fibered if its complement is a fiber bundle over the circle.
Case 3: \( P(-1, -1, 2m, p, q) \)

Claim 3.

Any \( P(-1, -1, 2m, p, q) \) is not fibered with \( m > 1 \) and \( 3 \leq p \leq q \): odd.

We use an algorithm to decide which pretzel knot is fibered developed by [Gabai].

This completes the proof of Thm 1.
Remark \textbf{(necessity of Clm.1)}

On hyperbolic knots in \(S^3\),

- by \textbf{Cyclic Surgery Theorem} [CGLS],
  all cyclic surgeries must be \textbf{integral}.

- however,
  by \textbf{Finite Surgery Theorem} [Boyer-Zhang],
  finite surgeries are only shown to be \textbf{half-integral}.
  (Conjecture: it is actually integral)
3. On alternating Montesinos knots
An **alternating diagram**

\[ = \text{the crossings alternate under, over, under, over, as you travel along the knot.} \]

A knot is **alternating** if it has an alternating diagram.

\[ P(3, 5, 8) = \]

\[ P(-3, 5, 8) = \]
Seifert fibered surgery on alternating Montesinos knots

**Theorem 2 [I.-Jong-Mizushima]**

\( K \): hyperbolic alternating Montesinos knot

Then \( K \) admits no Seifert fibered surgery

**Remark:**

A Montesinos knot is alternating if and only if its reduced Montesinos diagram is alternating.
The new result of Wu

Theorem 2 follows from:

**Theorem [Wu ’09]**

If $M(p_1/q_1, p_2/q_2, p_3/q_3)$ with $q_1 \leq q_2 \leq q_3$ admits an atoroidal Seifert fibered surgery, then $q_1 = 2$, $(q_1, q_2) = (3, 3)$, or $(q_1, q_2, q_3) = (3, 4, 5)$.

[arXiv:0910.4882] Ying-Qing Wu

Immersed surfaces and Seifert fibered surgery on Montesinos knots together with...
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Theorem 3

Theorem 3 [I.-Jong-Mizushima].

\( K \): hyperbolic alternating Montesinos knot.

If \( K \) admits SF surgery, then \( K = P(p, q, r) \)

with \( p, q, r \geq 3 \): mutually distinct odds.

Proposition

Let \( K \) be an hyperbolic alternating knot.

Then \( K \) admits No Tor & SF surgery.

Also, [I.-Jong] showed;

any hyperbolic Montesinos knot admits

NO toroidal & Seifert fibered surgery.


Toroidal Seifert fibered surgeries on Montesinos knots
Outline of Proof of Thm 3.

Let $K$ be an alternating hyperbolic Montesinos knot.

**Proposition 1**

If $K$ admits SF surgery, then either

(i) $K = P(p, q, r)$ with $p, q, r \geq 3$: odd,
(ii) $K = P(3, 3, 2n)$ with $n \geq 1$, or
(iii) $K = M(\frac{1}{3}, \frac{1}{3}, \frac{2m-1}{2m})$ with $m \geq 2$.

This is proved also based on Delman’s work:


“Constructing essential laminations and taut foliations which survive all Dehn surgeries”
Outline of Pr. of Prop. 1

[Delman] showed that $K$ admits an **essential lamination** (introduced by [Gabai-Oertel]) in its exterior **surviving essential** after all non-trivial surgeries (called **persistent lamination**).

Also [Brittenham] showed that “certain” (called **genuine**) essential lamination cannot exist in a SF manifold.

Outline of Proof (Proposition 1)

1) Check whether the Delman’s lami. is genuine,
2) if not, construct “new” one, which is genuine.
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Proposition 2

(1) \( K \neq P(p, q, q) \) with odd \( p, q \geq 3 \).

(2) \( K \neq P(2n, q, q) \) with \( n \geq 1 \) and odd \( q \geq 3 \).

(3) \( K \neq M(\frac{1}{3}, \frac{1}{3}, \frac{2m-1}{2m}) \) with \( m \geq 2 \).

In the following, we prove Proposition 2 (1).

Set \( K = P(p, q, q) \). Suppose that \( K(r) \) is SF.

Fact 1 [I.]

\( r \) must be an integer. i.e., \( r \in \mathbb{Z} \).
Montesinos trick

Then, by so-called Montesinos trick, 

\[ K(r) \cong 2\text{-fold branched cover of } S^3 \text{ along } L(p, q, r). \]

Remark

\[ L(p, q, r) \text{ is a knot (} \iff r : \text{ odd}) \]

or a 2-component link (\( \iff r : \text{ even} \)).
Montesinos trick

Fact 2

\( K(r) : \text{SF} \Rightarrow L(p, q, r) : \text{a Montesinos or Seifert link.} \)

Seifert link = the exterior admits a Seifert fibration.

Lemma A.

\( L(p, q, r) \) is not a Montesinos link.

Lemma B.

\( L(p, q, r) \) is not a Seifert link.

These give a contradiction.

Then we complete the proof of Proposition 2 (1).

From now on, we will show Lemma A.
Lemma A (link)

[Proof of Lem. A \((r: \text{ even}, \text{i.e.}, L(p, q, r): 2\text{-comp.})\)]

Claim 1.

\(r: \text{ even} \Rightarrow L(p, q, r) \neq \text{ Montesinos link.}\)

[Proof] Suppose: \(r: \text{ even} \& L(p, q, r): \text{ Montesinos link}\)

\[L(p, q, r) = T_{2,q} \cup T_{2,p+q} \Rightarrow b(L(p, q, r)) \geq 4\]

On the other hand, by Facts 3 & 4, \(b(L(p, q, r)) \leq 3\).

\(\Rightarrow \Leftarrow \text{ contradiction.} \quad \square\)

Facts

(Fact 3) \(L(p, q, r) \neq \text{ Montesinos link of } l > 3.\)

(Fact 4) \(L: \text{ Montesinos link with length } l \Rightarrow b(L) \leq l.\)
Lemma A (knot)

[Proof of Lem. A ($r$: odd, i.e., $L(p,q,r)$: knot)]

Here we introduce the alternation number.

**alternative number [Kawauchi]**

$\mathcal{A} = \{\text{alternating links}\} \ (\ni \text{trivial links}).$

$d_G(\cdot, \cdot) : \text{the Gordian distance}.$

$$\text{alt}(L) = \min_{L' \in \mathcal{A}} d_G(L, L').$$

Remark

- $\text{alt}(K) \leq u(K)$
- $\forall n \in \mathbb{N}, \exists K \text{ s.t. } \text{alt}(K) = n. \ [\text{Abe}], \ [\text{Kawauchi}]$
Fact 5. [Abe-Jong-Kishimoto]

\[ \text{alt}(L) \leq 1 \quad \text{for any Montesinos link } L. \]

Fact 6. [Rasmussen], [Abe]

For a knot \( K \), \( \text{alt}(K) \geq \frac{|s(K) - \sigma(K)|}{2} \)
where \( s(K) \): the Rasmussen invariant of \( K \), and
\( \sigma(K) \): the signature of \( K \)
with \( \sigma(\text{right-handed trefoil}) = 2. \)

Claim 2.

\[ \text{alt}(L(p, q, r)) \geq 2. \]

Claim 2 guarantees \( L(p, q, r) \neq \text{Montesinos knot}. \)
Namely, \( \text{Claim 2 is true } \Rightarrow \text{Lemma A is true} \).
Rasmussen invariant

Based on a generalization of Khovanov homology developed by Lee, $H_{\text{Lee}}(L)$

$$H_k := \{ x \in H_{\text{Lee}}(L) \mid x = [x] \text{ for } \exists x \in C_k \}$$

$q(x) = k \iff x \in H_k \& x \not\in H_{k+1}$ (called q-grading)

**Def. [Rasmussen invariant]**

$$s(K) := \min \{q(x) \mid x \in H_{\text{Lee}}(K), x \neq 0\} + 1$$

Khovanov homology: a categorification of Jones polynomial

**Thm. [Khovanov]**

$$\sum_{i,j} (-1)^i q^j \rank H^{i,j}(L) = J_L(q)$$
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[Proof of Claim 2.]

Fact 7 [I.], [Lackenby-Meyerhoff]

$K$ : hyperbolic knot. $r, r' \in \mathbb{Z}$.

$r, r'$-surgery is exceptional $\Rightarrow \Delta(r, r') \leq 8$.

SubClaim 1

$-8 \leq r \leq 8$.

[Proof] $K(0)$ contains essential torus $\Rightarrow$

0-surgery is exceptional. By Fact 7, $-8 \leq r \leq 8$. $\square$

Remark

$L(p, q, r)$ is a positive knot for $-8 \leq r \leq 8$. 

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SubClaim 2

\[ s(L(p, q, r)) = s(L(p, q - 2, r)) + 8. \]

[Proof] Direct calculation by using

Fact 8 [Rasmussen]

For a positive knot \( K \) with a positive diagram \( D \),

\[ s(K) = c(D) - o(D) + 1 \]

where \( c(D) \): the number of crossings of \( D \),

\( o(D) \): the number of Seifert circles of \( D \).

In particular, \( \sigma(L(p, q, r)) \leq s(L(p, q, r)) = 2g_*(K) \).
SubClaim 3

\[ \sigma(L(p, q, r)) \leq \sigma(L(p, q - 2, r)) + 4. \]

[Proof] \( L(p, q - 2, r) \) is obtained from \( L(p, q, r) \) by a single \#-unknotting operation.

Applying Fact 9, we complete the proof of SubClaim 3.

Fact 9 [H. Murakami]

\[ K_0 : \text{2-component link} \Rightarrow \sigma(K) \leq \sigma(K') + 4. \]
By Facts 6, 8 and SubClaims 1, 2, 3,

\[ 2 \text{alt}(L(p, q, r)) \geq s(L(p, q, r)) - \sigma(L(p, q, r)) \geq (s(L(p, q - 2, r)) + 8) - (\sigma(L(p, q - 2, r)) + 4) = s(L(p, q - 2, r)) - \sigma(L(p, q - 2, r)) + 4 \geq 4. \]

This completes [Proof of Claim 2] \[ \text{alt}(L(p, q, r)) \geq 2 \] □

By Fact 5, \( L(p, q, r) \neq \) Montesinos knot

\( (p, q \geq 3, -8 \leq r \leq 8) \).

Consequently, we completes [Proof of Lemma A].

Namely, \( L(p, q, r) \neq \text{Montesinos} \). □
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[Proof of Lem B.] \( L(p, q, r) \neq \text{Seifert link} \)

Fact 10

A Seifert link with (\# of comp.) \( \leq 2 \) is either

(i) a torus knot,

(ii) a torus link,

(iii) (a torus knot) \( \cup \) (the core of the torus).

By using various knot invariant

(e.g., determinant, genus, braid index),

it is shown that \( L(p, q, r) \neq \text{Seifert link} \) in the above.