Crosscap numbers of pretzel knots

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§ 1. Introduction

Any knot in $S^3$ bounds an **orientable** subsurface in $S^3$; called a **Seifert surface**.
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Any knot in $S^3$ bounds an orientable subsurface in $S^3$; called a Seifert surface.

Any knot in $S^3$ also bounds a non-orientable subsurface in $S^3$. (consider checkerboard surfaces)
The genus of a knot $K$ is defined to be the minimal genus of a Seifert surface for $K$. 
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**Definition. [Clark, ’78]**

The crosscap number $\gamma(K)$ of a knot $K$ is defined to be the minimal 1st betti number of a non-orientable surface spanning $K$ in $S^3$.

For completeness we define $\gamma(K) = 0$ if and only if $K$ is the unknot.
Example

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In fact, $\gamma(K) = 2$: by

Proposition [Clark]

$\gamma(K) = 1$ iff $K$ is a $(2, n)$-cabled knot.
Known results:

- $\gamma(7_4) = 3$
  (Murakami-Yasuhara, ’95)

- Formula for torus knots  (Teragaito, ’04)

- Algorithm for two-bridge knots
  (Hirasawa-Teragaito, )
§ 2. Result

We determine \( \gamma(K) \) for pretzel knots.
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$K$ is a knot $\Rightarrow$

(a) some $p_i$ is even, and the others are odd, or,
(b) $n$ is odd, and $p_1, p_2, \ldots, p_n$ are all odd.
Observations:

For (type a), $\gamma(K) \leq n - 1$

\[
\left( \text{consider naturally spanned \ non-orientable surface} \right);
\]

\[
\left( \text{consider naturally spanned Seifert surface, and} \right);
\]
Observations:

For (type a), $\gamma(K) \leq n - 1$

(consider naturally spanned non-orientable surface)

For (type b), $\gamma(K) \leq n$

(consider naturally spanned Seifert surface, and adding a small half-twist)
Theorem.

Let $K$ be a pretzel knot $P(p_1, \cdots, p_n)$. If the length $n \geq 2$, then

$$\gamma(K) = \begin{cases} n - 1 & \text{(type a)} \\ n & \text{(type b)} \end{cases}$$

Remark:

¢ For $n = 2$, $K$ is $(2, k)$-cabled, and $\gamma(K) = 1$.
¢ For $n = 1$, $K$ is trivial, and $\gamma(K) = 0$.
¢ For (type b), $\gamma(K) = 2g(K) + 1$ holds.
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§ 3. Outline of Proof

Let $K$ be a pretzel knot. Actually we prove:
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Proposition

Any essential surface $F$ for $K$ satisfies

$$\frac{-\chi}{\#s}(F) \geq \begin{cases} 
  n - 3 & \text{(type a)} \\
  n - 2 & \text{(type b)} 
\end{cases}$$  \hspace{1cm} (1)$$

Moreover, if the equality holds,

$F$ fails to be a non-ori spanning surface.
\( \chi \): the Euler characteristic of \( F \)

\( \#s \): the number of sheets

i.e. minimal number of \( \partial F \cap \text{meridian} \)
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**Proposition ⇒ Theorem:**

Given a non-ori spanning surface \( F \) attaining \( \gamma(K) \).

If \( F \) is essential, Prop ⇒ Thm , directly.

Otherwise, by boundary-compression, \( F \Rightarrow \) essential surface \( F' \) with smaller \( -\chi \).

Then, Prop ⇒ Thm , directly again. \( \square \)
Key to prove Proposition:

⇒ Hatcher - Oertel’s Algorithm

A. Hatcher and U. Oertel,
Boundary slopes for Montesinos knots,

They gave an Algorithm to list up
all boundary slopes for a given Montesinos knot.
Any essential surface $F$ for $K$ corresponds to a set of edgepaths in the left diagram.

We call such a set of edgepaths the edgepath system corresponding to $F$. 
Remark:

Edgepath system $\Rightarrow$ properly embedded surface, called candidate surface.
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For candidate surfaces, $\exists$ formula of $\frac{-\chi}{\#s}$
(Implicitly, in H-O. Also see Dunfield’s program)
Candidate surf. are classified into type I, II, or III.
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For type II and III;
it is easy to check Ineq. (1) holds.
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it is easy to check Ineq.(1) holds.

For type I;
we can (easily) check Ineq.(1) holds except for the following special cases:
$P(-2, 3, 3), P(-2, 3, 5), P(3, 3, n), P(3, 5, 5)$
These cases can be checked individually.
Formula of $\chi_s$ for type I surface

$$\chi_s(\Gamma) = \sum_{i=1}^{n} \left\{ \begin{array}{ll} 0 & \text{(if } \Gamma_i \text{ is constant)} \\ |\Gamma_i| & \text{(otherwise)} \end{array} \right\} + n_{\text{const}} - n + \left( n - 2 - \sum_{\Gamma_i \text{ is Constant Edgepaths}} \frac{1}{q_i} \right) \frac{1}{1 - u}.$$