Lens spaces which are unobtainable by surgery on knots

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Lens spaces

Lens space $L(p, q)$: the 3-manifold obtained by $p/q$-surgery on the trivial knot in $S^3$. 
Problem

Decide the lens spaces obtainable by surgery on non-trivial knots in $S^3$. 
Known results

- (Moser)
  $M$ : obtained by $p/q$-surgery on a $(r, s)$-torus knot. $M$ is a lens space $\Rightarrow M = L(|p|, qs^2)$.

- (Bleiler-Litherland, Wang, Wu)
  $M$ : obtained by surgery on a satellite knot $K$. $M$ is a lens space $\Rightarrow \begin{cases} K : \text{the (}2pq \pm 1, 2\text{)-cable} \\
M = L(4pq \pm 1, 4q^2). \end{cases}$
• (Hirasawa-Shimokawa) 
  $L(2p, 1)$ is unobtainable by surgery on strongly invertible knots.

• (Goda-Teragaito) 
  Lens spaces are unobtainable by surgery on genus one hyperbolic knots.

• (Kronheimer-Mrowka-Ozsváth-Szabó) 
  $L(p, 1)$ is unobtainable by $p$-surgery on knots.
Gordon’s conjecture

\( M \) : obtained by non-trivial surgery on a non-trivial knot.

\( M \) is a lens space \( \Rightarrow \) \( M \) is obtained by Berge’s surgery on a doubly primitive knot.
Doubly primitive knots

$(V_1, V_2; S)$ : a genus two Heegaard splitting of $S^3$.

$K$ : a simple loop on $S$.

$K$ is a **doubly primitive knot** if $K$ represents a free generator of both $\pi_1(V_1)$ and $\pi_1(V_2)$. 
‘Conjecture’ (Ichihara-Teragaito)

Lens spaces containing Klein bottles are unobtainable by surgery on non-torus knots.
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Lens spaces containing Klein bottles are unobtainable by surgery on non-torus knots.

Theorem

Lens spaces containing Klein bottles are unobtainable by Berge’s surgery on non-torus doubly primitive knots.
Dual knots

$K$: a knot in a 3-manifold $N$.

$N' := E(K; N) \cup V$, where $V$ is a filling solid torus.

$K^*$: a core loop of $V$.

Then $K^*$ is the dual knot of $K$. 
**Dual knots**

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Then \( K^* \) is the **dual knot** of \( K \).

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**Theorem** (Berge)

\( K \): a doubly primitive knot.

\[ S^3 \xrightarrow{\text{Berge's surgery}} L(p, q) \]

\[ \cup \]

\[ K \xrightarrow{\text{dual}} K^* \]

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Dual knots

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Theorem (Berge)

\( K \): a doubly primitive knot.

\[
\begin{align*}
S^3 & \xrightarrow{\text{Berge's surgery}} L(p, q) \\
\cup & \\
K & \xleftarrow{\text{}} K^* = K(L(p, q); u)
\end{align*}
\]
$K(L(p, q); u)$
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\( t^u_i \): a simple arc in \( D_i \) joining \( P_0 \) to \( P_u \).
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$K(L(p, q); u) := t_1^u \cup t_2^u$
Basic sequences

For positive coprime integers $p$ and $q$, 
$\{qj \pmod{p}\}_{j=1}^{p}$ is called a basic sequence.
Observation

$L(p, 2)$ is unobtainable by Berge’s surgery on non-torus doubly primitive knots.
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*K*: a doubly primitive knot.

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S^3 \cup K \xrightarrow{\text{Berge's surgery}} L(p, 2) \cup K^* = K(L(p, 2); u)
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**Observation**

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\[ K: \text{a doubly primitive knot.} \]

\[ S^3 \cup K \xrightarrow{\text{Berge’s surgery}} L(p, 2) \downarrow \cup K^* = K(L(p, 2); u) \]

\[ \{2j \pmod{p}\}_{j=1}^p : 2, 4, \ldots, p-1, 1, 3, \ldots, p-2, 0 \]
Case 1. $u \equiv 0 \pmod{2}$.

$$\{2j \pmod{p}\}_{j=1}^p : 2, \ldots, u, \ldots, p - 1, 1, \ldots, p - 2, 0$$
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\{2j \pmod{p}\}_{j=1}^{p} : 2, \ldots, u, \ldots, p - 1, 1, \ldots, p - 2, 0
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Case 2. \( u \equiv 1 \pmod{2} \).
\[
\left\{ 2j \pmod{p} \right\}_{j=1}^{p} : 2, \ldots, p-1, 1, \ldots, u, \ldots, p-2, 0
\]
Case 2. $u \equiv 1 \pmod{2}$.

$\{2j \pmod{p}\}_{j=1}^{p} : 2, \ldots, p-1, 1, \ldots, u, \ldots, p-2, 0$
$K(L(p, q); u)$ with $S^3$-surgery

There is an algorithm to determine whether a given lens space is obtainable by Berge’s surgery on a doubly primitive knot.
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**Step 1** Consider a 2-bridge link $S(p, q)$. 
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Step 2  Attach a band to $S(p, q)$ as follows.
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$$=: \mathcal{K}_0$$
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Step 2  Attach a band to $S(p, q)$ as follows.

\[
=: \mathcal{K}_1
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Step 1  Consider a 2-bridge link $S(p, q)$.

Step 2  Attach a band to $S(p, q)$ as follows.

Step 3  Check whether $\mathcal{K}_0$ or $\mathcal{K}_1$ is a trivial knot.
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Conclusion

$K(L(p, q); u)$ admits integral $S^3$-surgery

$\iff \mathcal{K}_0$ or $\mathcal{K}_1$ is a trivial knot.