Gromov hyperbolicity of a variation of the Gordian complex

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joint work with
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§1. Introduction

$\mathcal{K} := \{ \text{knots in the 3-sphere} \}$. $K, K' \in \mathcal{K}$. $K \triangleleft\triangleleft K' \iff K$ can be deformed into $K'$ by $\triangleleft\triangleleft$ once.

**Gordian distance** $d^\times(\cdot, \cdot)$

\[
d^\times(K, K') := \min\{ n \mid K = K_0 \triangleleft\triangleleft \cdots \triangleleft\triangleleft K_n = K' \}.
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**Gordian graph** \( G^K \)

- \{ vertices \} = \( \mathcal{K} \).
- \( \exists \) an edge between \( K \) and \( K' \) \( \iff \) \( d^K(K, K') = 1 \).

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**Remark 1.1**

\( G^K \) = the 1–skelton of the Gordian complex

introduced by [Hirasawa-Uchida ’02].
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$G^x$ = the 1–skelton of the Gordian complex introduced by [Hirasawa-Uchida '02].
Proposition 1.2 [Gambaudo-Ghys ’05]. The Gordian graph $G^x$ is not Gromov hyperbolic.
(We introduce the Gromov hyperbolicity later.)
\[ \lambda : \text{a local move on knots}. \]

\[ K \xleftarrow{\lambda} K' \iff K \text{ can be deformed into } K' \text{ by } \lambda \text{ once.} \]

**λ-Gordian distance** \[ d^\lambda(\cdot, \cdot) \]
\[ d^\lambda(K, K') := \min\{n \mid K = K_0 \xleftarrow{\lambda} \cdots \xleftarrow{\lambda} K_n = K' \}. \]

**λ-Gordian graph** \[ G^\lambda \]

- \{ vertices \} = \mathcal{K}.
- ∃ an edge between \( K \) and \( K' \) \iff \( d^\lambda(K, K') = 1 \).
$\lambda$: a local move on knots.

$K \xrightarrow{\lambda} K' \iff K$ can be deformed into $K'$ by $\lambda$ once.

**$\lambda$-Gordian distance $d^{\lambda} (\cdot, \cdot)$**

$$d^{\lambda}(K, K') := \min\{ n \mid K = K_0 \xrightarrow{\lambda} \cdots \xrightarrow{\lambda} K_n = K' \}.$$ 

**$\lambda$-Gordian graph $G^{\lambda}$**

- $\{ \text{vertices} \} = \mathcal{K}$.
- $\exists$ an edge between $K$ and $K' \iff d^{\lambda}(K, K') = 1$.

**Problem 1.3.**

For a given local move $\lambda$, detect whether $G^{\lambda}$ is Gromov hyperbolic or not.
§2. Gromov hyperbolicity

Assumption on graphs

Graphs are connected & each edge has length 1.
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**Assumption on graphs**

Graphs are connected & each edge has length 1.

\( \Gamma \) : a graph. \( N(\gamma, \varepsilon) \) : \( \varepsilon \)-nbd. of \( \gamma \subset \Gamma \).

**\( \delta \)-thin**

\( T \) : a triangle with sides \( s_1, s_2, s_3 \) in \( \Gamma \).

\( T \) is \( \delta \)-thin \( \iff \) \( s_i \subset N(s_j \cup s_k, \delta) \) for different \( i, j, k \).

\[ s_3 \subset N(s_1 \cup s_2, \delta). \]
Hyperbolicity

\[ \Gamma \text{ is } \delta\text{-hyperbolic (or Gromov hyperbolic)} \Leftrightarrow \]
any geodesic triangle in \( \Gamma \) is \( \delta \)-thin for \( \delta > 0 \).

(A geodesic triangle \( \Leftrightarrow \) each side is a geodesic.)
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Example 2.1

- Any tree is 0-hyperbolic.
- $\mathbb{R}^2$ is not Gromov hyperbolic.
- $\mathbb{H}^2$ is $\frac{\log 3}{2}$-hyperbolic.
- A graph with finite diameter $r$ is $r$-hyperbolic.
Hyperbolicity

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**Example 2.2** [Masur-Minsky ’99]

For a surface \( S \), the curve complex \( \mathcal{C}(S) \) is Gromov hyperbolic. The constant \( \delta \) depends on \( S \).
§3. $(\iota, \lambda)$-Gordian graph

$\iota$ : a knot invariant.

$K \sim_{\iota} K' : \iff \iota(K) = \iota(K')$ for $K, K' \in \mathcal{K}$.

$[K]_{\iota}$ : the equivalence class of $K$ w.r.t. $\sim_{\iota}$.

$\mathcal{K}_{\iota} := \{ [K]_{\iota} \mid K \in \mathcal{K} \}$
§3. \((\iota, \lambda)\)-Gordian graph

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\(\mathcal{K}_\iota := \{ [K]_\iota \mid K \in \mathcal{K} \}\)

\(-\)Gordian graph \(G^\lambda_\iota\)

- \(\{\text{vertices}\} = \mathcal{K}_\iota\).
- \(\exists\) an edge between \([K]_\iota\) and \([K']_\iota\) \(\iff\)
  \[ \exists J \in [K]_\iota, \exists J' \in [K']_\iota \text{ s.t. } d^\lambda(J, J') = 1. \]

\(d^\lambda_\iota\) : the metric on \(G^\lambda_\iota\).
\( \nabla_K \): the Conway polynomial of a knot \( K \).

\[
(\nabla_K = 1 + a_2 z^2 + \cdots + a_{2n} z^{2n}.)
\]

Proposition 3.1.

The \( (\nabla, x) \)-Gordian graph \( G_x \nabla \) has diameter 2.

Proof.

Any \( [K] \) \( \nabla \) contains a knot with unknotting number 1 \([Kondo],[Sakai]\). Thus, \( \text{diam} \ G_x \nabla \leq 2 \). On the other hand, \( \text{d} \ x \nabla ([3], [4]) = 2 \) \([Kawauchi]\). \( \square \)

Corollary 3.2.

The graph \( G_x \nabla \) is Gromov hyperbolic.

Proof.

Any graph with finite diameter is Gromov hyperbolic. \( \square \)
\( \nabla_K \): the Conway polynomial of a knot \( K \).

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**Proposition 3.1.** The \((\nabla, x)\)-Gordian graph \( G^x_{\nabla} \) has diameter 2.

Proof. Any \([K]_{\nabla}\) contains a knot with unknotting number 1 [Kondo], [Sakai]. Thus, \( \text{diam } G^x_{\nabla} \leq 2 \).
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The graph \( G^x_{\nabla} \) is Gromov hyperbolic.

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**Main Theorem** [Ichihara-J.].

The graph $G \triangleleft$ is Gromov hyperbolic.

**Remark 3.3.**

The diameter of $G \triangleleft$ is infinite.
A local picture of $G^\Lambda$

\[ a_m \ (m \geq 4) \]

\[ a_2 \]

\[ K \]
§4. \((\nabla, \Delta)\)-Gordian distance \(d^{\nabla}\)

**Lemma 4.1** [Okada '90].

Let \(K\) and \(K'\) be knots with \(d^{\Delta}(K, K') = 1\). Then we have \(a_2(K) - a_2(K') = \pm 1\).
§4. $(\nabla, \Delta)$-Gordian distance $d\nabla$

**Lemma 4.1** [Okada '90].

Let $K$ and $K'$ be knots with $d\Delta(K, K') = 1$. Then we have $a_2(K) - a_2(K') = \pm 1$.

**Notation:** $[K] = [K]_\nabla$.

**Lemma 4.2.**

For any $[K], [K']$, we have the following.

- $d\nabla([K], [K']) \geq |a_2(K) - a_2(K')|$.
- $d\nabla([K], [K']) \equiv |a_2(K) - a_2(K')| \pmod{2}$.

**Recall:** For $K_1, K_2 \in [K]$, we have $\nabla K_1 = \nabla K_2$. 
Lemma 4.3.

For $[K] \neq [K']$ with $a_2 = a_2(K)$, $a'_2 = a_2(K')$,

(1) $a_2 = a'_2 \Rightarrow d\wedge([K], [K']) = 2$.

(2) $|a_2 - a'_2| \geq 2 \Rightarrow d\wedge([K], [K']) = |a_2 - a'_2|$.

(3) $|a_2 - a'_2| = 1, \Rightarrow d\wedge([K], [K']) = 1$ or $3$.

The proof are achieved by constructing knots which satisfy given conditions. We give the proof later.
§4. Proof of Main Theorem

\( V_n := \{ [K] \in \mathcal{K}_\nabla \mid a_2(K) = n \} \).

\( S_n \): the subgraph of \( G_\nabla \) induced by \( V_n \cup V_{n\pm 1} \).

**Lemma 5.1.**

For \( [K] \in \mathcal{K}_\nabla \) with \( a_2(K) = n \), \( N([K], 3) \supset S_n \).

(The proof is immediately obtained by Lemma 4.3.)
Main Theorem [Ichihara-J.].

The graph $G\triangleleft$ is Gromov hyperbolic.

Proof of Main Theorem

There are several cases.

We only show the theorem for a particular case.

Other cases are shown in a similar way.
$T$: a geodesic triangle with sides $s_1$, $s_2$, $s_3$.

Let

\[
s_1 = \overline{x_0x_1} \cup \overline{x_1x_2} \cup \cdots \cup \overline{x_{p-1}x_p},
\]

\[
s_2 = \overline{y_0y_1} \cup \overline{y_1y_2} \cup \cdots \cup \overline{y_{q-1}y_q},
\]

\[
s_3 = \overline{z_0z_1} \cup \overline{z_1z_2} \cup \cdots \cup \overline{z_{r-1}z_r},
\]

where $x_0, \ldots, x_p, y_0, \ldots, y_q, z_0, \ldots, z_r \in \mathcal{K}_\nabla$ with

\[
x_0 = x = z_r, \quad y_0 = y = x_p, \quad \text{and} \quad z_0 = z = y_q.
\]
The figure is an example of $T$ ($p = 4$, $q = 4$, $r = 8$).

**To show:** $T$ is 3-thin, namely,

$$N(s(x, y) \cup s(y, z), 3) \supset s(z, x),$$
$$N(s(y, z) \cup s(z, x), 3) \supset s(x, y),$$
$$N(s(z, x) \cup s(x, y), 3) \supset s(y, z).$$
By Lemma 5.1, \( N(y_j, 3) \supseteq z_{q-j+1}z_{q-j} , z_{q-j}z_{q-j-1} \) for \( j = 1, \ldots, q - 1 \).
By Lemma 5.1, $N(y_j, 3) \supset \overline{z_{q-j+1}z_{q-j}}$, $\overline{z_{q-j}z_{q-j-1}}$ for $j = 1, \cdots, q - 1$. Thus, we have

$$N(s(y, z), 3) \supset \overline{z_qz_{q-1}} \cup \cdots \cup \overline{z_1z_0}.$$
Similarly, \( N(x_j, 3) \supset \overline{z_{r-j+1}} \overline{z_{r-j}} , \overline{z_{r-j}} \overline{z_{r-j-1}} \) for \( j = 1, \ldots, p - 1 \).
Similarly, $N(x_j, 3) \supseteq \bar{z}_{r-j+1} \bar{z}_{r-j}, \bar{z}_{r-j} \bar{z}_{r-j-1}$ for $j = 1, \ldots, p - 1$. Thus, we have

$$N(s(x, y), 3) \supseteq \bar{z}_r \bar{z}_{r-1} \cup \cdots \cup \bar{z}_{p+1} \bar{z}_p.$$
Similarly, \( N(x_j, 3) \supset z_{r-j+1} \cup z_{r-j} \cup \cdots \cup z_{p+1} \cup z_p \) for \( j = 1, \cdots, p - 1 \). Thus, we have
\[
N(s(x, y), 3) \supset z_r \cup z_{r-1} \cup \cdots \cup z_{p+1} \cup z_p.
\]
Therefore we have \( N(s(x, y) \cup s(y, z), 3) \supset s(z, x) \).
Remaining two conditions

- \( N(s(y, z) \cup s(z, x), 3) \supset s(x, y) \) and
- \( N(s(z, x) \cup s(x, y), 3) \supset s(y, z) \)

are shown by the similar argument.

Therefore the geodesic triangle \( T \) is 3-thin. \( \square \)
Lemma 4.3.

For \([K] \neq [K']\) with \(a_2 = a_2(K), a'_2 = a_2(K')\),

1. \(a_2 = a'_2 \Rightarrow d_{\Delta}(\mathbb{K}, [K'], [K]) = 2\).

2. \(|a_2 - a'_2| \geq 2 \Rightarrow d_{\Delta}(\mathbb{K}, [K'], [K']) = |a_2 - a'_2|\).

3. \(|a_2 - a'_2| = 1, \Rightarrow d_{\Delta}(\mathbb{K}, [K'], [K']) = 1 \text{ or } 3\).
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For \([K] \neq [K']\) with \(a_2 = a_2(K), a'_2 = a_2(K')\),

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Proof. Let \(K_m\) be the twist knot. Then we have

\(\nabla_{K_m} = 1 + mz^2\), and \(d\Delta(K_{m+1}, K_m) = 1\).
Let $K_{\pm}(\alpha_1, \ldots, \alpha_n)$ be knots as following.

By [Murakami], [Yamada], we have

$$\nabla K_{\pm}(\alpha_1, \ldots, \alpha_n) = 1 + \sum_{i=1}^{n} (-1)^{i-1} \alpha_i z^{2i},$$

and $d^\Delta(K_{\pm}(\alpha_1, \ldots, \alpha_n), K_{\alpha_1 \pm 1}) = 1.$
For \([K] \neq [K'] \in \mathcal{K}_\nabla\), let

\[
\nabla_K = 1 + \sum_{i=1}^{n} a_{2i} z^{2i}, \quad \nabla_{K'} = 1 + \sum_{i=1}^{m} a'_{2i} z^{2i},
\]

\[
J_+ = K_+(a_2, \ldots, (-1)^{n-1}a_{2n}), \text{ and}
\]

\[
J'_\pm = K_\pm(a'_2, \ldots, (-1)^{m-1}a'_2m).
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\[J'_\pm = K_\pm(a'_2, \ldots, (-1)^{m-1}a'_{2m}).\]

(1) \(a_2 = a'_2 \Rightarrow d_{\triangle}([K],[K']) = 2.\)

- We have \(d_{\triangle}([K],[K']) \geq 2\) by Lemma 4.2.
For $[K] \neq [K'] \in \mathcal{K}_\nabla$, let

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\]

\[
J_+ = K_+(a_2, \ldots, (-1)^{n-1} a_{2n}), \quad \text{and}
\]

\[
J'_+ = K_+(a'_2, \ldots, (-1)^{m-1} a'_{2m}).
\]

\[
(1) \quad a_2 = a'_2 \Rightarrow d_{\nabla}([K], [K']) = 2.
\]

- We have $d_{\nabla}([K], [K']) \geq 2$ by Lemma 4.2.
- We have $d_{\nabla}([K], [K']) \leq 2$ by the sequence of knots $J_+, K_{a_2+1}, J'_+$. 

\[
(d^\Delta(J_+, J'_+) \leq 2, \quad \nabla_{J_+} = \nabla_K, \quad \text{and} \quad \nabla_{J'_+} = \nabla_{K'}).\]
\(2\) \(|a_2 - a'_2| \geq 2 \Rightarrow d_{\Delta}(\{K\}, \{K'\}) = |a_2 - a'_2|\).

We may assume that \(a'_2 \geq a_2 + 2\).

- We have \(d_{\Delta}(\{K\}, \{K'\}) \geq a'_2 - a_2\) by Lemma 4.2.
- We have \(d_{\Delta}(\{K\}, \{K'\}) \leq a'_2 - a_2\) by the sequence of knots \(J_+, K_{a_2+1}, \ldots, K_{a'_2-1}, J'_-\).
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(3) \(|a_2 - a'_2| = 1, \Rightarrow d_{\triangle}(\{K\}, \{K'\}) = 1\ or\ 3\).

We may assume that \(a'_2 = a_2 + 1\).

- We have \(d_{\triangle}(\{K\}, \{K'\}) \equiv 1\ mod\ 2\ by\ Lemma\ 4.2\).
- We have \(d_{\triangle}(\{K\}, \{K'\}) \leq 3\ by\ the\ sequence\ of\ knots\ \(J_+, K_{a_2+1}, K_{a_2}, J'_-\).\)

\(\square\) (Lemma 4.3)