L-space surgery and twisting operation

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A knot in the 3–sphere is called an L-space knot if it admits a nontrivial Dehn surgery yielding an L-space, i.e. a rational homology 3–sphere with the smallest possible Heegaard Floer homology. Given a knot K, take an unknotted circle c and twist K n times along c to obtain a twist family $\{K_n\}$. We give a sufficient condition for $\{K_n\}$ to contain infinitely many L-space knots. As an application we show that for each torus knot and each hyperbolic Berge knot K, we can take c so that the twist family $\{K_n\}$ contains infinitely many hyperbolic L-space knots. We also demonstrate that there is a twist family of hyperbolic L-space knots each member of which has tunnel number greater than one.

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1 Introduction

Heegaard Floer theory (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) associates a group $\widehat{\mathrm{HF}}(M,\mathbf{t})$ to a closed, orientable spin^c 3–manifold $(M,\mathbf{t})$. The direct sum of $\widehat{\mathrm{HF}}(M,\mathbf{t})$ for all spin^c structures is denoted by $\widehat{\mathrm{HF}}(M)$. A rational homology 3–sphere M is called an *L-space* if $\widehat{\mathrm{HF}}(M,\mathbf{t})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for all spin^c structure $\mathbf{t} \in Spin^c(M)$. Equivalently, the dimension $\dim_{\mathbb{Z}/2\mathbb{Z}}\widehat{\mathrm{HF}}(M)$ is equal to the order $|H_1(M;\mathbb{Z})|$. A knot K in the 3–sphere S^3 is called an *L-space knot* if the result $K(r)$ of r–surgery on K is an L-space for some non-zero integer r, and the pair (K,r) is called an *L-space surgery*. The class of L-spaces includes lens spaces (except $S^2 \times S^1$), and more generally, 3–manifolds with elliptic geometry [47, Proposition 2.3]. Since the trivial knot, nontrivial torus knots and Berge knots [6] admit nontrivial surgeries yielding lens spaces, these are fundamental examples of L-space knots. For the mirror image K^* of K, $K^*(-r)$ is orientation reversingly homeomorphic to $K(r)$. So if $K(r)$ is an L-space, then $K^*(-r)$ is also an L-space [47, p.1288]. Hence if K is an L-space knot, then so is K^*.

Let K be a nontrivial L-space knot with a positive L-space surgery, then Ozsváth and Szabó [48, Proposition 9.6] ([25, Lemma 2.13]) prove that r–surgery on K results in an L-space if and only if $r \geq 2g(K)-1$, where $g(K)$ denotes the genus of K. This result, together with Thurston's hyperbolic Dehn surgery theorem [51, 52, 4, 49, 7], shows that each hyperbolic L-space knot, say a hyperbolic Berge knot, produces infinitely many hyperbolic L-spaces by Dehn surgery.

On the other hand, there are some strong constraints for L-space knots:

- The non-zero coefficients of the Alexander polynomial of an L-space knot are ± 1 and alternate in sign [47, Corollary 1.3].
- An L-space knot is fibered [43, Corollary 1.2]([44]); see also [19, 29].
- An L-space knot is prime [31, Theorem 1.2].

Note that these conditions are not sufficient. For instance, 10_{132} satisfies the above conditions, but it is not an L-space knot; see [47].

As shown in [25, 26], some satellite operations keep the property of being L-space knots. In the present article, we consider if some suitably chosen twistings also keep the property of being L-space knots. Given a knot K, take an unknotted circle c which bounds a disk intersecting K at least twice. Then performing n–twist, i.e. $(-1/n)$–surgery along c, we obtain another knot K_n. Then our question is formulated as:

Question 1.1 Which knots K admit an unknotted circle c such that n–twist along c converts K into an L-space knot K_n for infinitely many integers n? Furthermore, if K has such a circle c, which circles enjoy the desired property?

Example 1.2 Let K be a pretzel knot $P(-2,3,7)$ and take an unknotted circle c as in Figure 1.1. Then following Ozsváth and Szabó [47] K_n is an L-space knot if $n \geq -3$ and thus the twist family $\{K_n\}$ contains infinity many L-space knots. Note that this family, together with a twist family $\{T_{2n+1,2}\}$, comprise all Montesinos L-space knots; see [33] and [3].

In this example, it turns out that c becomes a Seifert fiber in the lens space $K(19)$ (cf. Example 4.3). We employed such a circle for relating Seifert fibered surgeries in [13]. A pair (K,m) of a knot K in S^3 and an integer m is a *Seifert surgery* if $K(m)$ has a Seifert fibration; we allow the fibration to be degenerate, i.e. it contains an exceptional fiber of index 0 as a degenerate fiber. See [13, 2.1] for details. The definition below enables us to say that c is a seiferter for the Seifert (lens space) surgery $(K,19)$.

Definition 1.3 (**seiferter** [13]) Let (K,m) be a Seifert surgery. A knot c in $S^3 - N(K)$ is called a *seiferter* for (K,m) if c satisfies the following:

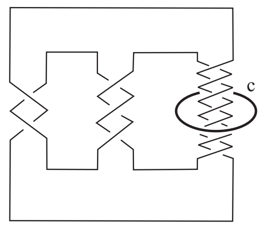

Figure 1.1: A knot K_n obtained by n–twist along c is an L-space knot if $n \geq -3$.

- c is a trivial knot in S^3.
- c becomes a fiber in a Seifert fibration of $K(m)$.

As remarked in [13, Convention 2.15], if c bounds a disk in $S^3 - K$, then we do not regard c as a seiferter. Thus for any seiferter c for (K, m), $S^3 - \mathrm{int}N(K \cup c)$ is irreducible.

Let (K, m) be a Seifert surgery with a seiferter c. There are two cases according as c becomes a fiber in a non-degenerate Seifert fibration of $K(m)$ or c becomes a fiber in a degenerate Seifert fibration of $K(m)$. In the former case, for homological reasons, the base surface is the 2–sphere S^2 or the projective plane $\mathbb{R}P^2$. Suppose that c is a fiber in a non-degenerate Seifert fibration of $K(m)$ over the 2–sphere S^2. Then in the following we assume that $K(m)$ contains at most three exceptional fibers and if there are three exceptional fibers, then c is an exceptional fiber. We call such a seiferter a *seiferter for a small Seifert fibered surgery* (K, m). To be precise, the images of K and m after n–twist along c should be denoted by $K_{c,n}$ and $m_{c,n}$, but for simplicity, we abbreviate them to K_n and m_n respectively as long as there is no confusion.

Theorem 1.4 *Let c be a seiferter for a small Seifert fibered surgery (K, m). Then (K_n, m_n) is an L-space surgery for an infinite interval of integers n if and only if the result of $(m, 0)$–surgery on $K \cup c$ is an L-space.*

In remaining cases, it turns out that every seiferter enjoys the desired property in Question 1.1.

Theorem 1.5 *Let c be a seiferter for (K, m) which become a fiber in a Seifert fibration of $K(m)$ over $\mathbb{R}P^2$. Then (K_n, m_n) is an L-space surgery for all but at most one integer n_0 with $(K_{n_0}, m_{n_0}) = (O, 0)$. Hence K_n is an L-space knot for all integers n.*

Let us turn to the case where c is a (degenerate or non-degenerate) fiber in a degenerate Seifert fibration of $K(m)$. Recall from [13, Proposition 2.8] that if $K(m)$ has a degenerate Seifert fibration, then it is a lens space or a connected sum of two lens spaces such that each summand is neither S^3 nor $S^2 \times S^1$. The latter 3–manifold will be simply referred to as a *connected sum of two lens spaces*, which is an L-space [50, 8.1(5)] ([45]).

Theorem 1.6 *Let c be a seiferter for (K,m) which becomes a (degenerate or non-degenerate) fiber in a degenerate Seifert fibration of $K(m)$.*

(1) *If $K(m)$ is a lens space, then (K_n, m_n) is an L-space surgery, hence K_n is an L-space knot, for all but at most one integer n.*

(2) *If $K(m)$ is a connected sum of two lens spaces, then (K_n, m_n) is an L-space surgery, hence K_n is an L-space knot, for any integer $n \geq -1$ or $n \leq 1$.*

Following Greene [23, Theorem 1.5], if $K(m)$ is a connected sum of two lens spaces, then K is a torus knot $T_{p,q}$ or a cable of a torus knot $C_{p,q}(T_{r,s})$, where $p = qrs \pm 1$. We may assume $p, q \geq 2$ by taking the mirror image if necessary. The next theorem is a refinement of Theorem 1.6(2).

Theorem 1.7 *Let c be a seiferter for $(K,m) = (T_{p,q}, pq)$ or $(C_{p,q}(T_{r,s}), pq)$ ($p = qrs \pm 1$). We assume $p, q \geq 2$. Then a knot K_n obtained from K by n–twist along c is an L-space knot for any $n \geq -1$. Furthermore, if the linking number l between c and K satisfies $l^2 \geq 2pq$, then K_n is an L-space knot for all integers n.*

In the above theorem, even when $\ell^2 < 2pq$, K_n ($n < -1$) may be an L-space knot; see [41].

In Sections 5, 6 and 7 we will exploit seiferter technology developed in [13, 11, 12] to give a partial answer to Question 1.1. Even though Theorem 1.7 treats a special kind of Seifert surgeries, it offers many applications. In particular, Theorem 1.7 enables us to give new families of L-space twisted torus knots. See Section 5 for the definition of twisted torus knots $K(p,q;r,n)$ introduced by Dean [10].

Theorem 1.8 **(L-space twisted torus knots)** (1) *The following twisted torus knots are L-space knots for all integers n.*

- $K(p,q;p+q,n)$ *with* $p,q \geq 2$
- $K(3p+1, 2p+1; 4p+1, n)$ *with* $p > 0$
- $K(3p+2, 2p+1; 4p+3, n)$ *with* $p > 0$

(2) *The following twisted torus knots are L-space knots for any $n \geq -1$.*

- $K(p, q; p-q, n)$ *with* $p, q \geq 2$
- $K(2p+3, 2p+1; 2p+2, n)$ *with* $p > 0$

Theorem 1.8 has the following corollary, which asserts that every nontrivial torus knot admits twistings desired in Question 1.1.

Corollary 1.9 *For any nontrivial torus knot $T_{p,q}$, we can take an unknotted circle c so that n–twist along c converts $T_{p,q}$ into an L-space knot K_n for all integers n. Furthermore, $\{K_n\}_{|n|>3}$ is a set of mutually distinct hyperbolic L-space knots.*

For the simplest L-space knot, i.e. the trivial knot O, we can strengthen Corollary 1.9 as follows.

Theorem 1.10 (**L-space twisted unknots**) *For the trivial knot O, we can take infinitely many unknotted circles c so that n–twist along c changes O into a nontrivial L-space knot $K_{c,n}$ for any non-zero integer n. Furthermore, $\{K_{c,n}\}_{|n|>1}$ is a set of mutually distinct hyperbolic L-space knots.*

Using a relationship between Berge's lens space surgeries and surgeries yielding a connected sum of two lens spaces, we can prove:

Theorem 1.11 (**L-space twisted Berge knots**) *For any hyperbolic Berge knot K, there is an unknotted circle c such that n–twist along c converts K into a hyperbolic L-space knot K_n for infinitely many integers n.*

In Section 8 we consider the tunnel number of L-space knots. Recall that the *tunnel number* of a knot K in S^3 is the minimum number of mutually disjoint, embedded arcs connecting K such that the exterior of the resulting 1–complex is a handlebody. Hedden's cabling construction [25], together with [40], enables us to obtain an L-space knot with tunnel number greater than 1. Actually Baker and Moore [3] have shown that for any integer N, there is an L-space knot with tunnel number greater than N. However, L-space knots with tunnel number greater than one constructed above are all satellite (non-hyperbolic) knots and they ask:

Question 1.12 ([3]) Is there a non-satellite, L-space knot with tunnel number greater than one?

Examining knots with Seifert surgeries which do not arise from primitive/Seifert-fibered construction given by [16], we prove the following which answers the question in the positive.

Theorem 1.13 *There exist infinitely many hyperbolic L-space knots with tunnel number greater than one.*

Each knot in the theorem is obtained from a trefoil knot $T_{3,2}$ by alternate twisting along two seiferters for the lens space surgery $(T_{3,2}, 7)$.

In Section 9 we will discuss further questions on relationships between L-space knots and twisting operation.

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2 Seifert fibered L-spaces

Let M be a rational homology 3–sphere which is a Seifert fiber space. For homological reasons, the base surface of M is either S^2 or $\mathbb{R}P^2$. In the latter case, Boyer, Gordon and Watson [8, Proposition 5] prove that M is an L-space. Now assume that the base surface of M is S^2. Following Ozsváth and Szabó [46, Theorem 1.4] if M is an L-space, then it carries no taut foliation, in particular, it carries no horizontal (i.e. transverse) foliation. Furthermore, Lisca and Stipsicz [34, Theorem 1.1] prove that the converse also does hold. Therefore a Seifert fibered rational homology 3–sphere M over S^2 is an L-space if and only if it does not admit a horizontal foliation. Note that if M does not carry a horizontal foliation, then it is necessarily a rational homology 3–sphere. In fact, if $|H_1(M;\mathbb{Z})| = \infty$, then M is a surface bundle over the circle [27, VI.34], [24], and hence it has a horizontal foliation. On the other hand, Eisenbud-Hirsh-Neumann

[14], Jankins-Neumann [28] and Naimi [42] gave a necessary and sufficient conditions for a Seifert fibered 3-manifold to carry a horizontal foliation. Combining them we have Theorem 2.1 below. See also [9, Theorem 5.4]; we follow the convention of Seifert invariants in [9, Section 4].

For ordered triples (a_1, a_2, a_3) and (b_1, b_2, b_3), we write $(a_1, a_2, a_3) < (b_1, b_2, b_3)$ (resp. $(a_1, a_2, a_3) \leq (b_1, b_2, b_3)$) if $a_i < b_i$ (resp. $a_i \leq b_i$) for $1 \leq i \leq 3$, and denote by $(a_1, a_2, a_3)^*$ the ordered triple $(\sigma(a_1), \sigma(a_2), \sigma(a_3))$, where σ is a permutation such that $\sigma(a_1) \leq \sigma(a_2) \leq \sigma(a_3)$.

Theorem 2.1 ([46, 34, 14, 28, 42]) *A Seifert fiber space $S^2(b, r_1, r_2, r_3)$ ($b \in \mathbb{Z}$, $0 < r_i < 1$) is an L-space if and only if one of the following holds.*

(1) $b \geq 0$ *or* $b \leq -3$.

(2) $b = -1$ *and there are no relatively prime integers* a, k *such that* $0 < a \leq k/2$ *and* $(r_1, r_2, r_3)^* < (1/k, a/k, (k-a)/k)$.

(3) $b = -2$ *and there are no relatively prime integers* $0 < a \leq k/2$ *such that* $(1 - r_1, 1 - r_2, 1 - r_3)^* < (1/k, a/k, (k-a)/k)$.

For our purpose, we consider the following problem:

Problem 2.2 Given an integer b and rational numbers $0 < r_1 \leq r_2 < 1$, describe rational numbers $-1 \leq r \leq 1$ for which $S^2(b, r_1, r_2, r)$ is an L-space.

We begin by observing:

Lemma 2.3 *Assume that* $0 < r_1 \leq r_2 < 1$.

(1) *If* $b \geq 0$ *or* $b \leq -3$, *then* $S^2(b, r_1, r_2, r)$ *is an L-space for any* $0 < r < 1$.

(2) *If* $r_1 + r_2 \geq 1$, *then* $S^2(-1, r_1, r_2, r)$ *is an L-space for any* $0 < r < 1$.

(3) *If* $r_1 + r_2 \leq 1$, *then* $S^2(-2, r_1, r_2, r)$ *is an L-space for any* $0 < r < 1$.

Proof of Lemma 2.3. The first assertion is nothing but Theorem 2.1(1).

Suppose for a contradiction that $S^2(-1, r_1, r_2, r)$ is not an L-space for some $0 < r < 1$. Then, by Theorem 2.1(2) we can take relatively prime integers a, k ($0 < a \leq k/2$) so that $(r_1, r_2, r)^* < (1/k, a/k, (k-a)/k)$. This then implies that $r_1 < a/k$ and $r_2 < (k-a)/k$. Hence $r_1 + r_2 < a/k + (k-a)/k = 1$, a contradiction. This proves (2).

To prove (3), assume for a contradiction that $S^2(-2, r_1, r_2, r)$ is not an L-space for some $0 < r < 1$. Then, by Theorem 2.1(3) we have relatively prime integers a, k $(0 < a \leq k/2)$ such that $(1-r_1, 1-r_2, 1-r)^* < (1/k, a/k, (k-a)/k)$. Thus we have $(1-r_2) < a/k$ and $(1-r_1) < (k-a)/k$. Thus $(1-r_1)+(1-r_2) < 1$, which implies $r_1 + r_2 > 1$, contradicting the assumption. □(Lemma 2.3)

Now let us prove the following, which gives an answer to Problem 2.2.

Proposition 2.4 *Assume that* $0 < r_1 \leq r_2 < 1$.

(1) *If* $b \leq -3$ *or* $b \geq 1$, *then* $S^2(b, r_1, r_2, r)$ *is an L-space for any* $-1 \leq r \leq 1$.

(2) *If* $b = -2$, *then there exists* $\varepsilon > 0$ *such that* $S^2(-2, r_1, r_2, r)$ *is an L-space for any* $-1 \leq r \leq \varepsilon$. *Furthermore, if* $r_1 + r_2 \leq 1$, *then* $S^2(-2, r_1, r_2, r)$ *is an L-space if* $-1 \leq r < 1$.

(3) *Suppose that* $b = -1$.

   (i) *If* $r_1 + r_2 \geq 1$, *then* $S^2(-1, r_1, r_2, r)$ *is an L-space for any* $0 < r \leq 1$.

   (ii) *If* $r_1 + r_2 \leq 1$, *then* $S^2(-1, r_1, r_2, r)$ *is an L-space for any* $-1 \leq r < 0$.

(4) *If* $b = 0$, *then there exists* $\varepsilon > 0$ *such that* $S^2(r_1, r_2, r)$ *is an L-space for any* $-\varepsilon \leq r \leq 1$. *Furthermore, if* $r_1 + r_2 \geq 1$, *then* $S^2(r_1, r_2, r)$ *is an L-space if* $-1 < r \leq 1$.

Proof of Proposition 2.4. If $r = 0, \pm 1$, then $S^2(b, r_1, r_2, r)$ is a lens space.

Claim 2.5 *Suppose that* r *is an integer. Then the lens space* $S^2(b, r_1, r_2, r)$ *is* $S^2 \times S^1$ *if and only if* $b + r = -1$ *and* $r_1 + r_2 = 1$. *In particular, if* $b + r \neq -1$, *then* $S^2(b, r_1, r_2, r)$ *is an L-space.*

Proof of Claim 2.5. Recall that $H_1(S^2(a/b, c/d)) \cong \mathbb{Z}$ $(b, d \geq 1)$ if and only if $ad + bc = 0$, i.e. $a/b + c/d = 0$. Thus $S^2(b, r_1, r_2, r)$ is $S^2 \times S^1$ if and only if $b + r_1 + r_2 + r = 0$, i.e. $r_1 + r_2 = -b - r \in \mathbb{Z}$. Since $0 < r_i < 1$, we have $r_1 + r_2 = 1$ and $b + r = -1$. □(Claim 2.5)

We divide into two cases according as $0 \leq r \leq 1$ or $-1 \leq r \leq 0$.

Case I. $0 \leq r \leq 1$.

(i) If $b \geq 0$ or $b \leq -3$, then $S^2(b, r_1, r_2, r)$ is an L-space for any $0 < r < 1$ by Lemma 2.3(1). Since $b + r \neq -1$ for $r = 0, 1$, by Claim 2.5 $S^2(b, r_1, r_2, r)$ is an L-space for $r = 0, 1$. Hence $S^2(b, r_1, r_2, r)$ is an L-space for any $0 \leq r \leq 1$.

(ii) Suppose that $b = -1$. By Lemma 2.3(2), if $r_1 + r_2 \geq 1$, then $S^2(-1, r_1, r_2, r)$ is an L-space for any $0 < r < 1$. Since $S^2(-1, r_1, r_2, 1)$ is an L-space (Claim 2.5), $S^2(-1, r_1, r_2, r)$ is an L-space for any $0 < r \leq 1$.

(iii) Assume $b = -2$. Let us assume $0 < r \leq r_1$ so that $0 < 1 - r_2 \leq 1 - r_1 \leq 1 - r < 1$. Set $A = \{(k-a)/k \mid 1 - r_2 < 1/k,\ 1 - r_1 < a/k,\ 0 < a \leq k/2,\ a \text{ and } k \text{ are relatively prime integers}\}$. If $A = \emptyset$, i.e. there are no relatively prime integers a, k $(0 < a \leq k/2)$ such that $1 - r_2 < 1/k$, $1 - r_1 < a/k$, then $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 < r \leq r_1$ by Theorem 2.1. Suppose that $A \neq \emptyset$. Since there are only finitely many integers k satisfying $1 - r_2 < 1/k$, A consists of only finitely many elements. Let r_0 be the maximal element in A. If $0 < r \leq 1 - r_0$, then $r_0 \leq 1 - r < 1$, and hence there are no relatively prime integers a, k $(0 < a \leq k/2)$ satisfying $(1 - r_2, 1 - r_1, 1 - r) < (1/k, a/k, (k-a)/k)$. Put $\varepsilon = \min\{r_1, 1 - r_0\}$. Then $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 < r \leq \varepsilon$ by Theorem 2.1. Since $S^2(-2, r_1, r_2, 0)$ is an L-space (Claim 2.5), $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 \leq r \leq \varepsilon$. Furthermore, if we have the additional condition $r_1 + r_2 \leq 1$, then Lemma 2.3(3) improves the result so that $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 \leq r < 1$.

Case II. $-1 \leq r \leq 0$.

Note that $S^2(b, r_1, r_2, r) = S^2(b - 1, r_1, r_2, r + 1)$.

(i) If $b \geq 1$ or $b \leq -2$ (i.e. $b - 1 \geq 0$ or $b - 1 \leq -3$), then $S^2(b, r_1, r_2, r) = S^2(b - 1, r_1, r_2, r + 1)$ is an L-space for any $0 < r + 1 < 1$, i.e. $-1 < r < 0$ by Lemma 2.3(1). Since $b + r \neq -1$ for $r = -1, 0$, $S^2(b, r_1, r_2, r)$ is an L-space for $r = -1, 0$ (Claim 2.5). Thus $S^2(b, r_1, r_2, r)$ is an L-space for any $-1 \leq r \leq 0$.

(ii) If $b = 0$ (i.e. $b - 1 = -1$), then $S^2(0, r_1, r_2, r) = S^2(-1, r_1, r_2, r + 1)$. Let us assume $r_2 - 1 \leq r < 0$ so that $0 < r_1 \leq r_2 \leq r + 1 < 1$. Set $A = \{(k-a)/k \mid r_1 < 1/k,\ r_2 < a/k,\ 0 < a \leq k/2,\ a \text{ and } k \text{ are relatively prime integers}\}$. If $A = \emptyset$, then we can easily observe that for any r with $r_2 \leq r + 1 < 1$, $S^2(-1, r_1, r_2, r + 1)$ is an L-space (Theorem 2.1). Hence for any $r_2 - 1 \leq r < 0$, $S^2(0, r_1, r_2, r)$ is an L-space. Suppose that $A \neq \emptyset$. Since A is a finite set, we take the maximal element r_0 in A. If $r_0 \leq r + 1 < 1$ (i.e. $r_0 - 1 \leq r < 0$), then there are no relatively prime integers a, k $(0 < a \leq k/2)$ satisfying $(r_1, r_2, r + 1) < (1/k, a/k, (k-a)/k)$. Put $\varepsilon = \min\{1 - r_2,\ 1 - r_0\}$. Then $S^2(0, r_1, r_2, r) = S^2(-1, r_1, r_2, r + 1)$ is an L-space for any $-\varepsilon \leq r < 0$ (Theorem 2.1). Since $S^2(0, r_1, r_2, 0) = S^2(r_1, r_2)$ is an L-space (Claim 2.5), $S^2(0, r_1, r_2, r)$ is an L-space for any $-\varepsilon \leq r \leq 0$. Furthermore, if we have the additional condition $r_1 + r_2 \geq 1$, then Lemma 2.3(2) improves the result so that $S^2(r_1, r_2, r) = S^2(-1, r_1, r_2, r+1)$ is an L-space for any $-1 < r \leq 0$.

(iii) If $b = -1$ (i.e. $b-1 = -2$), then $S^2(-1, r_1, r_2, r) = S^2(-2, r_1, r_2, r+1)$. Assume that $r_1 + r_2 \leq 1$. Then Proposition 2.3(3), $S^2(-1, r_1, r_2, r) = S^2(-2, r_1, r_2, r+1)$ is an L-space for any $0 < r+1 < 1$, i.e. $-1 < r < 0$. Since Claim 2.5 shows that $S^2(-1, r_1, r_2, -1)$ is an L-space, $S^2(-1, r_1, r_2, r)$ is an L-space for any $-1 \leq r < 0$.

Combining Cases I and II, we obtain the result described in the proposition.

□(Proposition 2.4)

The next proposition shows that if $S^2(b, r_1, r_2, r_\infty)$ is an L-space for some rational number $0 < r_\infty < 1$, then we can find r near r_∞ so that $S^2(b, r_1, r_2, r)$ is an L-space.

Proposition 2.6 *Suppose that* $0 < r_1 \leq r_2 < 1$ *and* $S^2(b, r_1, r_2, r_\infty)$ *is an L-space for some rational number* $0 < r_\infty < 1$.

(1) *If* $b = -1$, *then* $S^2(-1, r_1, r_2, r)$ *is an L-space for any* $r_\infty \leq r \leq 1$.

(2) *If* $b = -2$, *then* $S^2(-2, r_1, r_2, r)$ *is an L-space for any* $-1 \leq r \leq r_\infty$.

Proof of Proposition 2.6. (1) Assume for a contradiction that $S^2(-1, r_1, r_2, r)$ is not an L-space for some r satisfying $r_\infty \leq r < 1$. By Theorem 2.1 we have relatively prime integers a, k $(0 < a \leq k/2)$ such that $(r_1, r_2, r)^* < (1/k, a/k, (k-a)/k)$. Since $r_\infty \leq r < 1$, $(r_1, r_2, r_\infty)^* \leq (r_1, r_2, r)^* < (1/k, a/k, (k-a)/k)$. Hence Theorem 2.1 shows that $S^2(-1, r_1, r_2, r_\infty)$ is not an L-space, a contradiction. Since $S^2(-1, r_1, r_2, 1) = S^2(r_1, r_2)$ is an L-space (Claim 2.5), $S^2(-1, r_1, r_2, r)$ is an L-space for any $r_\infty \leq r \leq 1$.

(2) Next assume for a contradiction that $S^2(-2, r_1, r_2, r)$ is not an L-space for some r satisfying $0 < r \leq r_\infty$. Then following Theorem 2.1 we have $(1-r_1, 1-r_2, 1-r)^* < (1/k, a/k, (k-a)/k)$ for some relatively prime integers a, k $(0 < a \leq k/2)$. Since $r \leq r_\infty$, we have $1 - r_\infty \leq 1 - r$, and hence $(1-r_1, 1-r_2, 1-r_\infty)^* \leq (1-r_1, 1-r_2, 1-r)^* < (1/k, a/k, (k-a)/k)$. This means $S^2(-2, r_1, r_2, r_\infty)$ is not an L-space, contradicting the assumption. Thus $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 < r \leq r_\infty$. Furthermore, as shown in Proposition 2.4(2), $S^2(-2, r_1, r_2, r)$ is an L-space if $-1 \leq r \leq \varepsilon$ for some $\varepsilon > 0$, so $S^2(-2, r_1, r_2, r)$ is an L-space for any $-1 \leq r \leq r_\infty$. □(Proposition 2.6)

We close this section with the following result which is the complement of Proposition 2.6.

Proposition 2.7 *Suppose that $0 < r_1 \leq r_2 < 1$ and $S^2(b, r_1, r_2, r_\infty)$ is not an L-space for some rational number $0 < r_\infty < 1$.*

(1) *If $b = -1$, then there exists $\varepsilon > 0$ such that $S^2(-1, r_1, r_2, r)$ is not an L-space for any $0 < r < r_\infty + \varepsilon$.*

(2) *If $b = -2$, then there exists $\varepsilon > 0$ such that then $S^2(-2, r_1, r_2, r)$ is an L-space for any $r_\infty - \varepsilon < r < 1$.*

Proof of Proposition 2.7. (1) Since $S^2(-1, r_1, r_2, r_\infty)$ is not an L-space, Theorem 2.1 shows that there are relatively prime integers a, k $(0 < a \leq k/2)$ such that $(r_1, r_2, r_\infty)^* < (1/k, a/k, (k-a)/k)$. Then clearly there exists $\varepsilon > 0$ such that for any $0 < r < r_\infty + \varepsilon$, we have $(r_1, r_2, r)^* < (1/k, a/k, (k-a)/k)$. Thus by Theorem 2.1 again $S^2(-1, r_1, r_2, r)$ is not an L-space for any $0 < r < r_\infty + \varepsilon$.

(2) Since $S^2(-2, r_1, r_2, r_\infty)$ is not an L-space, by Theorem 2.1 we have relatively prime integers a, k $(0 < a \leq k/2)$ such that $(1-r_1, 1-r_2, 1-r_\infty)^* < (1/k, a/k, (k-a)/k)$. Hence there exists $\varepsilon > 0$ such that if $0 < 1-r < 1-r_\infty+\varepsilon$, i.e. $r_\infty - \varepsilon < r < 1$, then $(1-r_1, 1-r_2, 1-r)^* < (1/k, a/k, (k-a)/k)$. Following Theorem 2.1 $S^2(-2, r_1, r_2, r)$ is not an L-space for any $r_\infty - \varepsilon < r < 1$. □(Proposition 2.7)

3 L-space surgeries and twisting along seiferters I – non-degenerate case

The goal in this section is to prove Theorems 1.4 and 1.5.

Let c be a seiferter for a small Seifert fibered surgery (K, m). The 3–manifold obtained by $(m, 0)$–surgery on $K \cup c$ is denoted by $M_c(K, m)$.

Proof of Theorem 1.4. First we prove the "if" part of Theorem 1.4. If $K(m)$ is a lens space and c is a core of the genus one Heegaard splitting, then $K_n(m_n)$ is a lens space for any integer n. Thus (K_n, m_n) is an L-space surgery for all $n \in \mathbb{Z}$ except when $K_n(m_n) \cong S^2 \times S^1$, i.e. K_n is the trivial knot and $m_n = 0$ [17, Theorem 8.1]. Since $(K_n, m_n) = (K_{n'}, m_{n'})$ if and only if $n = n'$ [13, Theorem 5.1], there is at most one integer n such that $(K_n, m_n) = (O, 0)$. Henceforth, in the case where $K(m)$ is a lens space, we assume that $K(m)$ has a Seifert fibration over S^2 with two exceptional fibers t_1, t_2, and c becomes a regular fiber in this Seifert fibration.

Let E be $K(m) - \mathrm{int}N(c)$ with a fibered tubular neighborhood of the union of two exceptional fibers t_1, t_2 and one regular fiber t_0 removed. Then E is a product circle bundle over the four times punctured sphere. Take a cross section of E such that $K(m)$ is expressed as $S^2(b, r_1, r_2, r_3)$, where the Seifert invariant of t_0 is $b \in \mathbb{Z}$, that of t_i is $0 < r_i < 1$ $(i = 1, 2)$, and that of c is $0 \le r_3 < 1$. Without loss of generality, we may assume $r_1 \le r_2$. Let s be the boundary curve on $\partial N(c)$ of the cross section so that $[s] \cdot [t] = 1$ for a regular fiber $t \subset \partial N(c)$. Let (μ, λ) be a preferred meridian-longitude pair of $c \subset S^3$. Then $[\mu] = \alpha_3[s] + \beta_3[t] \in H_1(\partial N(c))$ and $[\lambda] = -\alpha[s] - \beta[t] \in H_1(\partial N(c))$ for some integers $\alpha_3, \beta_3, \alpha$ and β which satisfy $\alpha_3 > 0$ and $\alpha\beta_3 - \beta\alpha_3 = 1$, where $r_3 = \beta_3/\alpha_3$. Now let us write $r_c = \beta/\alpha$, which is the slope of the preferred longitude λ of $c \subset S^3$ with respect to (s, t)–basis.

Claim 3.1 *$M_c(K, m)$ is a (possibly degenerate) Seifert fiber space $S^2(b, r_1, r_2, r_c)$; if $r_c = -1/0$, then it is a connected sum of two lens spaces.*

Proof of Claim 3.1. $M_c(K, m)$ is regarded as a 3–manifold obtained from $K(m)$ by performing λ–surgery along the fiber $c \subset K(m)$. Since $[\lambda] = -\alpha[s] - \beta[t]$, $M_c(K, m)$ is a (possibly degenerate) Seifert fiber space $S^2(b, r_1, r_2, r_c)$. If $\alpha = 0$, i.e. $r_c = -1/0$, then $M_c(K, m)$ has a degenerate Seifert fibration and it is a connected sum of two lens spaces. □(Claim 3.1)

Recall that (K_n, m_n) is a Seifert surgery obtained from (K, m) by twisting n times along c. The image of c after the n–twist along c is also a seiferter for (K_n, m_n) and denoted by c_n. We study how the Seifert invariant of $K(m)$ behaves under the twisting. We compute the Seifert invariant of c_n in $K_n(m_n)$ under the same cross section on E.

Since we have

$$\begin{pmatrix} [\mu] \\ [\lambda] \end{pmatrix} = \begin{pmatrix} \alpha_3 & \beta_3 \\ -\alpha & -\beta \end{pmatrix} \begin{pmatrix} [s] \\ [t] \end{pmatrix},$$

it follows that

$$\begin{pmatrix} [s] \\ [t] \end{pmatrix} = \begin{pmatrix} -\beta & -\beta_3 \\ \alpha & \alpha_3 \end{pmatrix} \begin{pmatrix} [\mu] \\ [\lambda] \end{pmatrix}.$$

Twisting n times along c is equivalent to performing $-1/n$–surgery on c. A preferred meridian-longitude pair (μ_n, λ_n) of $N(c_n) \subset S^3$ satisfies $[\mu_n] = [\mu] - n[\lambda]$ and $[\lambda_n] = [\lambda]$ in $H_1(\partial N(c_n)) = H_1(\partial N(c))$.

We thus have

$$\begin{pmatrix}[s]\\ [t]\end{pmatrix} = \begin{pmatrix}-\beta & -n\beta-\beta_3\\ \alpha & n\alpha+\alpha_3\end{pmatrix}\begin{pmatrix}[\mu_n]\\ [\lambda_n]\end{pmatrix},$$

and it follows that

$$\begin{pmatrix}[\mu_n]\\ [\lambda_n]\end{pmatrix} = \begin{pmatrix}n\alpha+\alpha_3 & n\beta+\beta_3\\ -\alpha & -\beta\end{pmatrix}\begin{pmatrix}[s]\\ [t]\end{pmatrix}.$$

Hence, the Seifert invariant of the fiber c_n in $K_n(m_n)$ is $(n\beta+\beta_3)/(n\alpha+\alpha_3)$, and $K_n(m_n) = S^2(b, r_1, r_2, (n\beta+\beta_3)/(n\alpha+\alpha_3))$.

Remark 3.2 Since $(n\beta+\beta_3)/(n\alpha+\alpha_3)$ converges to β/α when $|n|$ tends to ∞, $M_c(K,m)$ can be regarded as the limit of $K_n(m_n)$ when $|n|$ tends to ∞.

We divide into three cases: $r_c = -1/0$, $r_c \in \mathbb{Z}$ or $r_c \in \mathbb{Q}\setminus\mathbb{Z}$. Except for the last case, we do not need the assumption that $M_c(K,m)$ is an L-space.

Case 1. $r_c = \beta/\alpha = -1/0$.

Since $\alpha_3 > 0$ and $\alpha\beta_3 - \beta\alpha_3 = 1$, we have $\alpha_3 = 1$, $\beta = -1$. Hence $K_n(m_n)$ is a Seifert fiber space $S^2(b, r_1, r_2, (n\beta+\beta_3)/(n\alpha+\alpha_3)) = S^2(b, r_1, r_2, -n+\beta_3)$, which is a lens space for any $n \in \mathbb{Z}$. Following Claim 2.5 $S^2(b, r_1, r_2, -n+\beta_3)$ is an L-space if $n \neq b+\beta_3+r_1+r_2$. Thus (K_n, m_n) is an L-space surgery for all $n \in \mathbb{Z}$ except at most $n = b+\beta_3+r_1+r_2$.

Next suppose that $r_c = \beta/\alpha \neq -1/0$. Then the Seifert invariant of c_n is

$$f(n) = \frac{n\beta+\beta_3}{n\alpha+\alpha_3} = \frac{\beta}{\alpha} + \frac{\beta_3 - \frac{\beta}{\alpha}\alpha_3}{n\alpha+\alpha_3} = r_c + \frac{\beta_3 - r_c\alpha_3}{n\alpha+\alpha_3}.$$

Since $\alpha\beta_3 - \beta\alpha_3 = \alpha(\beta_3 - r_c\alpha_3) = 1$, α and $\beta_3 - r_c\alpha_3$ have the same sign.

Case 2. $r_c \in \mathbb{Z}$. We put $r_c = p$. Then we can write $S^2(b, r_1, r_2, r_c) = S^2(b+p, r_1, r_2)$.

(i) If $b \leq -p-3$ or $b \geq -p+1$, then Proposition 2.4(1) shows that $S^2(b, r_1, r_2, f(n)) = S^2(b+p, r_1, r_2, f(n)-p)$ is an L-space if $-1 \leq f(n)-p \leq 1$, i.e. $p-1 \leq f(n) \leq p+1$.

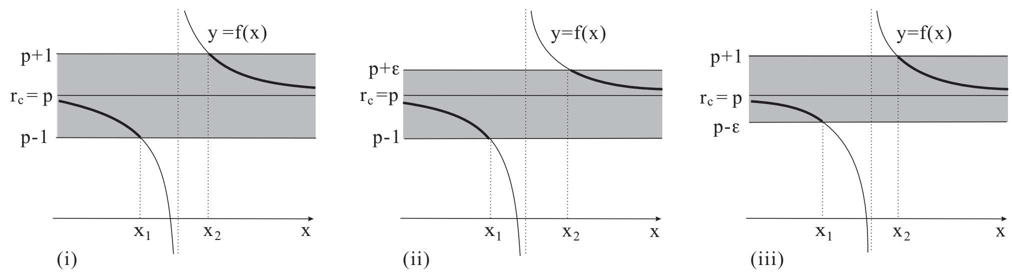

Figure 3.1: $f(x) = \frac{\beta x + \beta_3}{\alpha x + \alpha_3}$

Hence (K_n, m_n) is an L-space for all n but $n \in (x_1, x_2)$, where $f(x_1) = p - 1$ and $f(x_2) = p + 1$; see Figure 3.1(i).

(ii) If $b = -p - 2$, then it follows from Proposition 2.4(2), there is an $\varepsilon > 0$ such that $S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) = S^2(-2, r_1, r_2, f(n) - p)$ is an L-space if $-1 \leq f(n) - p \leq \varepsilon$. Hence (K_n, m_n) is an L-space except for only finitely many $n \in (x_1, x_2)$, where $f(x_1) = p - 1$, $f(x_2) = p + \varepsilon$; see Figure 3.1(ii).

(iii) Suppose that $b = -p - 1$. If $r_1 + r_2 \geq 1$ (resp. $r_1 + r_2 \leq 1$), then Proposition 2.4(3) shows that $S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) = S^2(-1, r_1, r_2, f(n) - p)$ is an L-space if $0 < f(n) - p \leq 1$ (resp. $-1 \leq f(n) - p < 0$). Hence (K_n, m_n) is an L-space for any integer $n \geq x_2$, where $f(x_2) = p + 1$ (resp. $n \leq x_1$, where $f(x_1) = p - 1$), see Figure 3.2.

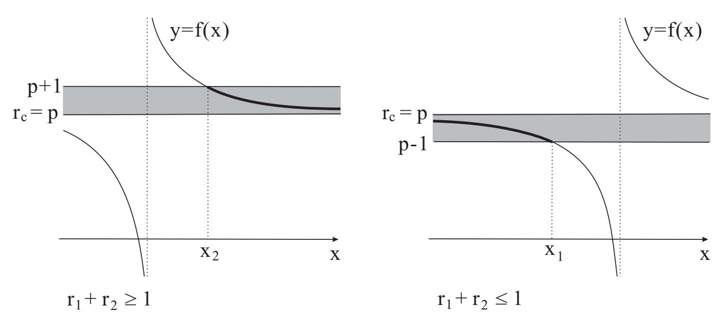

Figure 3.2: $f(x) = \frac{\beta x + \beta_3}{\alpha x + \alpha_3}$

(iv) If $b = -p$, then Proposition 2.4(4) shows that $S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) = S^2(r_1, r_2, f(n) - p)$ is an L-space if $-\varepsilon \leq f(n) - p \leq 1$, i.e.

$p-\varepsilon \le f(n) \le p+1$ for some $\varepsilon > 0$. Hence (K_n, m_n) is an L-space for all n but $n \in (x_1, x_2)$, where $f(x_1) = p-\varepsilon$ and $f(x_2) = p+1$; see Figure 3.1(iii).

Case 3. $r_c \in \mathbb{Q} \setminus \mathbb{Z}$ and $M_c(K,m) = S^2(b, r_1, r_2, r_c)$ is an L-space. We assume $p < r_c < p+1$ for some integer p. Then we have $S^2(b, r_1, r_2, r_c) = S^2(b+p, r_1, r_2, r_c - p)$, where $0 < r_c - p < 1$.

(i) If $b \le -p-3$ or $b \ge -p+1$, then Proposition 2.4(1) shows that $S^2(b, r_1, r_2, f(n)) = S^2(b+p, r_1, r_2, f(n)-p)$ is an L-space if $-1 \le f(n)-p \le 1$, i.e. $p-1 \le f(n) \le p+1$. Hence (K_n, m_n) is an L-space for all n but $n \in (x_1, x_2)$, where $f(x_1) = p-1$ and $f(x_2) = p+1$; see Figure 3.3(i).

(ii) Suppose that $b = -p-1$. Since $S^2(b, r_1, r_2, r_c) = S^2(b+p, r_1, r_2, r_c - p) = S^2(-1, r_1, r_2, r_c - p)$ is an L-space, by Proposition 2.6(1), $S^2(b, r_1, r_2, f(n)) = S^2(-1, r_1, r_2, f(n)-p)$ is an L-space if $r_c - p \le f(n) - p \le 1$ (i.e. $r_c \le f(n) \le p+1$). Hence (K_n, m_n) is an L-space for any $n \ge x_2$, where $f(x_2) = p+1$; see Figure 3.3(ii). (Furthermore, if $r_1 + r_2 \ge 1$, then by Proposition 2.4(3)(i), $S^2(-1, r_1, r_2, f(n)-p)$ is an L-space provided $0 < f(n) - p \le 1$, i.e. $p < f(n) \le p+1$. Hence (K_n, m_n) is an L-space surgery for any integer n except for $n \in [x_1, x_2)$, where $f(x_1) = p$ and $f(x_2) = p+1$.)

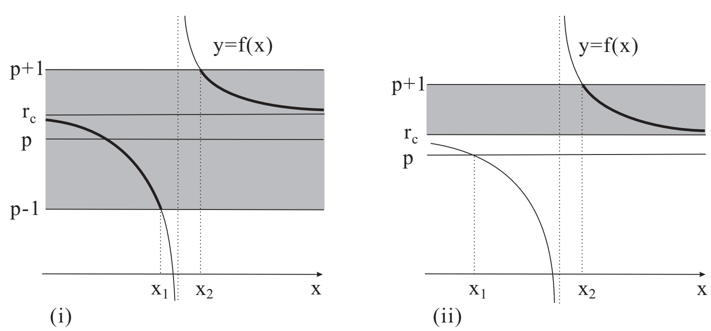

Figure 3.3: $f(x) = \frac{\beta x + \beta_3}{\alpha x + \alpha_3}$

(iii) Suppose that $b = -p-2$. Since $S^2(b, r_1, r_2, r_c) = S^2(b+p, r_1, r_2, r_c - p) = S^2(-2, r_1, r_2, r_c - p)$ is an L-space, following Proposition 2.6(2), $S^2(b, r_1, r_2, f(n)) = S^2(-2, r_1, r_2, f(n)-p)$ is an L-space if $-1 \le f(n) - p \le r_c - p$ (i.e. $p-1 \le f(n) \le r_c$). Hence (K_n, m_n) is an L-space for any $n \le x_1$, where $f(x_1) = p-1$; see Figure 3.4(i). (Furthermore, if $r_1 + r_2 \le 1$, then Proposition 2.4(2) shows that $S^2(-2, r_1, r_2, f(n)-p)$ is an L-space provided $-1 \le f(n) - p < 1$, i.e. $p-1 \le f(n) < p+1$. Hence (K_n, m_n)

is an L-space surgery for any integer n except for $n \in (x_1, x_2]$, where $f(x_1) = p-1$ and $f(x_2) = p+1$.)

(iv) If $b = -p$, then Proposition 2.4(4) shows that $S^2(b, r_1, r_2, f(n)) = S^2(b+p, r_1, r_2, f(n)-p) = S^2(r_1, r_2, f(n)-p)$ is an L-space if $-\varepsilon \le f(n)-p \le 1$, i.e. $p-\varepsilon \le f(n) \le p+1$ for some $\varepsilon > 0$. Hence (K_n, m_n) is an L-space for all n but $n \in (x_1, x_2)$, where $f(x_1) = p-\varepsilon$ and $f(x_2) = p+1$; see Figure 3.4(ii).

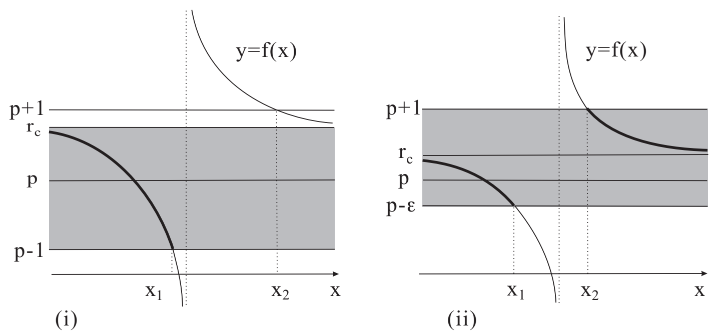

Figure 3.4: $f(x) = \frac{\beta x + \beta_3}{\alpha x + \alpha_3}$

Now let us prove the “only if" part of Theorem 1.4. We begin by observing:

Lemma 3.3 *$M_c(K, m)$ cannot be $S^2 \times S^1$, in particular, if $M_c(K, m)$ is a lens space, then it is an L-space.*

Proof of Lemma 3.3. Let w be the linking number between c and K. Then $H_1(M_c(K, m)) = \langle\ \mu_c,\ \mu_K \mid w\mu_c + m\mu_K = 0,\ w\mu_K = 0\ \rangle$, where μ_c is a meridian of c and μ_K is that of K. If $M_c(K, m) \cong S^2 \times S^1$, then $H_1(M_c(K, m)) \cong \mathbb{Z}$, and we have $w = 0$. Let us put $V = S^3 - \mathrm{int}N(c)$, which is a solid torus containing K in its interior; K is not contained in any 3–ball in V. Since $w = 0$, K is null-homologous in V. Furthermore, since c is a seiferter for (K, m), the result $V(K; m)$ of V after m–surgery on K has a (possibly degenerate) Seifert fibration. Then [13, Lemma 3.22] shows that the Seifert fibration of $V(K; m)$ is non-degenerate and neither a meridian nor a longitude of V is a fiber in $V(K; m)$, and the base surface of $V(K; m)$ is not a Möbius band. Since K is null-homologous in V, $V(K; m)$ is not a solid torus [18, Theorem 1.1], and hence, $V(K; m)$ has a Seifert fibration over the disk with at least two exceptional fibers. Then $M_c(K, m) = V(K; m) \cup N(c)$ is obtained by attaching $N(c)$ to $V(K; m)$ so that the meridian of $N(c)$ is identified with a meridian of V. Since a regular fiber on $\partial V(K; m)$ intersects a meridian of V, i.e. a meridian of $N(c)$ more than once, $M_c(K,m)$ is a Seifert fiber space over S^2 with at least three exceptional fibers. Therefore $M_c(K,m)$ cannot be $S^2 \times S^1$. This completes a proof. □(Lemma 3.3)

Suppose first that $K(m)$ is a lens space and c is a core of a genus one Heegaard splitting of $K(m)$. Then $V(K;m) = K(m) - \mathrm{int}N(c)$ is a solid torus and $M_c(K,m) = V(K;m) \cup N(c)$ is obviously a lens space. By Lemma 3.3 $M_c(K,m)$ is an L-space.

In the remaining case, as in the proof of the "if" part of Theorem 1.4, $M_c(K,m)$ has a form $S^2(b, r_1, r_2, r_c)$ $(0 < r_1 \le r_2 < 1)$.

Claim 3.4 *If $r_c = -1/0$ or $r_c \in \mathbb{Z}$, then $M_c(K,m)$ is an L-space.*

Proof of Claim 3.4. If $r_c = -1/0$, then $M_c(K,m) = S^2(b, r_1, r_2, -1/0)$ is a connected sum of two lens spaces. Since a connected sum of L-spaces is also an L-space [50, 8.1(5)] ([45]), $M_c(K,m)$ is an L-space. If $r_c \in \mathbb{Z}$, then $M_c(K,m)$ is a lens space, hence it is an L-space by Lemma 3.3. □(Claim 3.4)

Now suppose that $M_c(K,m)$ is not an L-space. Then by Claim 3.4 $r_c \in \mathbb{Q} \setminus \mathbb{Z}$. We write $r_c = r'_c + p$ so that $0 < r'_c < 1$ and $p \in \mathbb{Z}$. Then $M_c(K,m) = S^2(b, r_1, r_2, r_c) = S^2(b+p, r_1, r_2, r'_c)$. Since $M_c(K,m)$ is not an L-space, $b+p = -1$ or -2 (Theorem 2.1). It follows from Proposition 2.7 that there is an $\varepsilon > 0$ such that $K_n(m_n) = S^2(b, r_1, r_2, f(n)) = S^2(b+p, r_1, r_2, f(n)-p) = S^2(-1, r_1, r_2, f(n)-p)$ (resp. $S^2(-2, r_1, r_2, f(n)-p)$) is not an L-space if $0 < f(n)-p < r'_c+\varepsilon$, i.e. $p < f(n) < r_c+\varepsilon$ (resp. $r'_c - \varepsilon < f(n) - p < 1$, i.e. $r_c - \varepsilon < f(n) < p+1$). Hence there are at most finitely many integers n such that $K_n(m_n)$ is an L-space, i.e. (K_n, m_n) is an L-space surgery. See Figure 3.5.

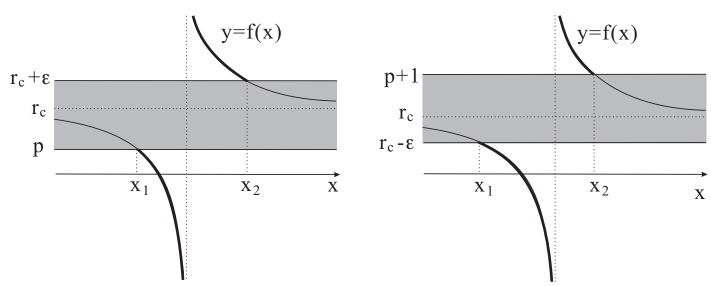

Figure 3.5: $f(x) = \frac{\beta x + \beta_3}{\alpha x + \alpha_3}$

This completes a proof of Theorem 1.4. □(Theorem 1.4)

Proof of Theorem 1.5. Note that $K_n(m_n)$ is a Seifert fiber space which admits a Seifert fibration over $\mathbb{R}P^2$, or $K_n(m_n)$ has $S^2 \times S^1$ as a connected summand according as c becomes a non-degenerate fiber, or a degenerate fiber in $K_n(m_n)$, respectively. In the former case, Boyer, Gordon and Watson [8, Proposition 5] prove that $K_n(m_n)$ is an L-space. In the latter case, $(K_n, m_n) = (O, 0)$ [17, Theorem 8.1], which is not an L-space surgery, but there is at most one such integer n [13, Theorem 5.1]. This completes a proof. □(Theorem 1.5)

Example 3.5 Let us consider a three component link $O \cup c_1 \cup c_2$ depicted in Figure 3.6. It is shown in [13, Lemma 9.26] that c_1, c_2 become fibers in a Seifert fibration of $O(0)$. Let A be an annulus in S^3 cobounded by c_1 and c_2. Performing $(-l)$–annulus twist along A, equivalently performing $(1/l + 3)$–, $(-1/l + 3)$–surgeries on c_1, c_2 respectively, we obtain a knot K_l given by Eudave-Muñoz [15]. Then, as shown in [15], $(K_l, 12l^2 - 4l)$ is a Seifert surgery such that $K_l(12l^2 - 4l)$ is a Seifert fiber space over $\mathbb{R}P^2$ with at most two exceptional fibers c_1, c_2 of indices $|l|$, $|-3l+1|$ for $l \neq 0$, where we use the same symbol c_i to denote the image of c_i after $(-l)$–annulus twist along A. Let c be one of c_1 or c_2. Then c is a seiferter for $(K_l, 12l^2 - 4l)$. Theorem 1.5 shows that a knot $K_{l,n}$ obtained from K_l by n–twist along c is an L-space knot for all integers n.

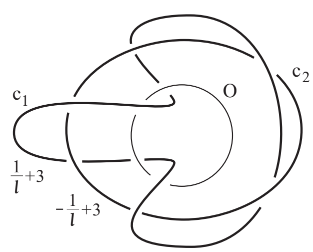

Figure 3.6: c_1 and c_2 become fibers in a Seifert fibration of $O(0)$.

4 L-space surgeries and twisting along seiferters II – degenerate case

In this section we will prove Theorem 1.6.

Proof of Theorem 1.6. Since $K(m)$ has a degenerate Seifert fibration, it is a lens space or a connected sum of two lens spaces [13, Proposition 2.8].

(1) $K(m)$ is a lens space with degenerate Seifert fibration.

Then there are at most two degenerate fibers in $K(m)$ [13, Proposition 2.8]. Assume that there are exactly two degenerate fibers. Then $(K,m) = (O,0)$ and the exterior of these two degenerate fibers is $S^1 \times S^1 \times [0,1]$. If c is a non-degenerate fiber, then $K_n(m_n)$ has $S^2 \times S^1$ as a connected summand for all integers n, and thus $(K_n, m_n) = (O,0)$ for all integers n [17, Theorem 8.1]. This contradicts [13, Theorem 5.1]. If c is one of the degenerate fibers, then (K_n, m_n) is a lens space, which is $S^2 \times S^1$ only when $(K_n, m_n) = (O,0) = (K_0, m_0)$, i.e. $n = 0$ [13, Theorem 5.1]. Thus (K_n, m_n) is an L-space surgery except when $n = 0$.

Suppose that $K(m)$ has exactly one degenerate fiber t_d. There are two cases to consider: $K(m) - \mathrm{int}N(t_d)$ is a fibered solid torus or has a non-degenerate Seifert fibration over the Möbius band with no exceptional fiber ([13, Proposition 2.8]). In either case, a meridian of t_d is identified with a regular fiber on $\partial(K(m) - \mathrm{int}N(t_d))$.

Assume that $K(m) - \mathrm{int}N(t_d)$ is a fibered solid torus. Suppose that c is a non-degenerate fiber. If c is a core of the solid torus, then $K(m) - \mathrm{int}N(c)$ is a solid torus and $K_n(m_n)$ is a lens space. Hence (K_n, m_n) is an L-space surgery except when $K_n(m_n) \cong S^2 \times S^1$, i.e. $(K_n, m_n) = (O,0)$. By [13, Theorem 5.1] there is at most one such integer n. If c is not a core in the fibered solid torus $K(m) - \mathrm{int}N(t_d)$, then $K_n(m_n)$ is a lens space ($\not\cong S^2 \times S^1$), a connected sum of two lens spaces, or a connected sum of $S^2 \times S^1$ and a lens space ($\not\cong S^3$, $S^2 \times S^1$). The last case cannot happen for homological reasons, and hence (K_n, m_n) is an L-space surgery. If c is the degenerate fiber t_d, then $K_n(m_n)$ is a lens space, and except for at most integer n_0 with $(K_{n_0}, m_{n_0}) = (O,0)$, (K_n, m_n) is an L-space surgery.

Next consider the case where $K(m) - \mathrm{int}N(t_d)$ has a non-degenerate Seifert fibration over the Möbius band. Then $(K,m) = (O,0)$; see [13, Proposition 2.8]. If c is a non-degenerate fiber, $K_n(m_n)$ has $S^2 \times S^1$ as a connected summand for all integers n. This implies that $(K_n, m_n) = (O,0)$ for all n [17, Theorem 8.1], contradicting [13, Theorem 5.1]. Thus c is a degenerate fiber, and $K_n(m_n)$ $(n \neq 0)$ is a Seifert fiber space over $\mathbb{R}P^2$ with at most one exceptional fiber, which has finite fundamental group. Hence for any non-zero integer n, (K_n, m_n) is an L-space [47, Proposition 2.3]. It follows that if c is a fiber in a degenerate Seifert fibration of a lens space $K(m)$, then (K,m) is an L-space surgery except for at most one integer n.

(2) $K(m)$ is a connected sum of two lens spaces.

It follows from [13, Proposition 2.8] that $K(m)$ has exactly one degenerate fiber t_d and $K(m) - \mathrm{int}N(t_d)$ is a Seifert fiber space over the disk with two exceptional fibers. Note that a meridian of t_d is identified with a regular fiber on $\partial(K(m) - \mathrm{int}N(t_d))$. We divide into two cases according as c is a non-degenerate fiber or a degenerate fiber.

(i) c is a non-degenerate fiber.

By [13, Corollary 3.21(1)] c is not a regular fiber. Hence c is an exceptional fiber, and $K_n(m_n)$ is a lens space ($\not\cong S^2 \times S^1$), a connected sum of two lens spaces, or a connected sum of $S^2 \times S^1$ and a lens space ($\not\cong S^3,\ S^2 \times S^1$). The last case cannot happen for homological reasons. Hence (K_n, m_n) is an L-space surgery for any integer n.

(ii) c is a degenerate fiber, i.e. $c = t_d$.

As in the proof of Theorem 1.4, let E be $K(m) - \mathrm{int}N(c)$ with a fibered tubular neighborhood of the union of two exceptional fibers t_1, t_2 and one regular fiber t_0 removed. Then E is a product circle bundle over the fourth punctured sphere. Take a cross section of E such that $K(m)$ has a Seifert invariant $S^2(b, r_1, r_2, 1/0)$, where the Seifert invariant of t_0 is $b \in \mathbb{Z}$, that of t_i is $0 < r_i < 1$ $(i = 1, 2)$, and that of c is $1/0$. We may assume that $r_1 \le r_2$. Let s be the boundary curve on $\partial N(c)$ of the cross section so that $[s] \cdot [t] = 1$ for a regular fiber $t \subset \partial N(c)$. Then $[\mu] = [t] \in H_1(\partial N(c))$ and $[\lambda] = -[s] - \beta[t] \in H_1(\partial N(c))$ for some integer β, i.e. we have:

$$\begin{pmatrix} [\mu] \\ [\lambda] \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\beta \end{pmatrix} \begin{pmatrix} [s] \\ [t] \end{pmatrix}$$

Let c_n be the image of c after n–twist along c. Then the argument in the proof of Theorem 1.4 shows that a preferred meridian-longitude pair (μ_n, λ_n) of $\partial N(c_n)$ has the expression:

$$\begin{pmatrix} [\mu_n] \\ [\lambda_n] \end{pmatrix} = \begin{pmatrix} n & n\beta + 1 \\ -1 & -\beta \end{pmatrix} \begin{pmatrix} [s] \\ [t] \end{pmatrix}$$

Thus $K_n(m_n) = S^2(b, r_1, r_2, (n\beta + 1)/n) = S^2(b + \beta, r_1, r_2, (n\beta + 1)/n - \beta) = S^2(b + \beta, r_1, r_2, 1/n)$ for non-zero integer n.

Claim 4.1 *$K_n(m_n)$ is an L-space for $n = 0, \pm 1$.*

Proof of Claim 4.1. Recall that $K_0(m_0) = K(m)$ is a connected sum of two lens spaces L_1 and L_2 such that $H_1(L_1) \cong \mathbb{Z}_{\alpha_1}$ and $H_1(L_2) \cong \mathbb{Z}_{\alpha_2}$, where $r_i = \beta_i/\alpha_i$. Thus $K_0(m_0)$ is an L-space. Since $K_{-1}(m_{-1})$ and $K_1(m_1)$ are lens spaces, it remains to show that they are not $S^2 \times S^1$. Assume for a contradiction that $K_1(m_1)$ or $K_{-1}(m_{-1})$ is $S^2 \times S^1$. Then Claim 2.5 shows that $r_1 + r_2 = 1$, hence $r_2 = \beta_2/\alpha_2 = (\alpha_1 - \beta_1)/\alpha_1$. Thus $\alpha_1 = \alpha_2$, and $H_1(K_0(m_0)) \cong \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2}$ is not cyclic, a contradiction. Hence neither $K_1(m_1)$ nor $K_{-1}(m_{-1})$ is $S^2 \times S^1$ and they are L-spaces. □(Claim 4.1)

(1) If $b + \beta \leq -3$ or $b + \beta \geq 1$, Proposition 2.4(1) shows that $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an L-space if $-1 \leq 1/n \leq 1$, i.e. $n \leq -1$ or $n \geq 1$. See Figure 4.1(i). Since $K_0(m_0)$ is also an L-space (Claim 4.1), $K_n(m_n)$ is an L-space for any integer n.

(2) If $b + \beta = -2$, Proposition 2.4(2) shows that there is an $\varepsilon > 0$ such that $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an L-space if $-1 \leq 1/n \leq \varepsilon$. Hence $K_n(m_n)$ is an L-space if $n \leq -1$ or $n \geq 1/\varepsilon$. See Figure 4.1(ii). This, together with Claim 4.1, shows that $K_n(m_n)$ is an L-space if $n \leq 1$ or $n \geq 1/\varepsilon$.

(3) Suppose that $b + \beta = -1$. Then Proposition 2.4(3) shows that if $r_1 + r_2 \geq 1$ (resp. $r_1 + r_2 \leq 1$), $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an L-space for any integer n satisfying $0 < 1/n \leq 1$ (resp. $-1 \leq 1/n < 0$), i.e. $n \geq 1$ (resp. $n \leq -1$). See Figure 4.1(i). Combining Claim 4.1, we see that $K_n(m_n)$ is an L-space for any $n \geq -1$ (resp. $n \leq 1$).

(4) If $b + \beta = 0$, then Proposition 2.4(4) shows that there is an $\varepsilon > 0$ such that $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an L-space if $-\varepsilon \leq 1/n \leq 1$. Hence $K_n(m_n)$ is an L-space if $n \geq 1$ or $n \leq -1/\varepsilon$. See Figure 4.1(iii). This, together with Claim 4.1, shows that $K_n(m_n)$ is an L-space if $n \geq -1$ or $n \leq -1/\varepsilon$.

This completes a proof of Theorem 1.6. □(Theorem 1.6)

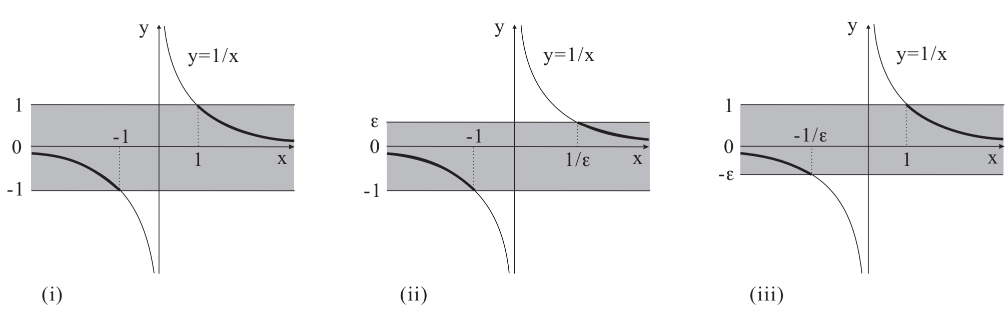

Figure 4.1: (i) $-1 \leq 1/n \leq 1$ if $n \leq -1$ or $n \geq 1$, (ii) $-1 \leq 1/n \leq \varepsilon$ if $n \leq -1$ or $n \geq 1/\varepsilon$, (iii) $-\varepsilon \leq 1/n \leq 1$ if $n \leq -1/\varepsilon$ or $n \geq 1$.

As shown by Greene [23, Theorem 1.5], if $K(m)$ is a connected sum of lens spaces, then K is a torus knot or a cable of a torus knot. More precisely, $(K,m)=(T_{p,q},pq)$ or $(C_{p,q}(T_{r,s}),pq)$, where $p=qrs\pm 1$. Note that $T_{p,q}(pq)=L(p,q)\sharp L(q,p)$ and $C_{p,q}(T_{r,s})(pq)=L(p,qs^2)\sharp L(q,\pm 1)$.

Let us continue to prove Theorem 1.7 which is a refinement of Theorem 1.6(2).

Proof of Theorem 1.7. In the following (K,m) is either $(T_{p,q},pq)$ or $(C_{p,q}(T_{r,s}),pq)$, where $p,q\geq 2$ and $p=qrs\pm 1$. If c becomes a non-degenerate fiber in $K(m)$, then as shown in the proof of Theorem 1.6, K_n is an L-space knot for any integer n. So we assume that c becomes a degenerate fiber in $K(m)$. Recall from Theorem 3.19(3) in [13] that the linking number l between c and K is not zero. Recall also that $K_n(m_n)$ is expressed as $S^2(b+\beta,r_1,r_2,1/n)=S^2(b+\beta,\beta_1/\alpha_1,\beta_2/\alpha_2,1/n)$, where $0<r_i=\beta_i/\alpha_i<1$ and $\alpha_i\geq 2$. See the proof of Theorem 1.6. Note that $\{\alpha_1,\alpha_2\}=\{p,q\}$, and $\alpha_1\alpha_2=pq\geq 6$.

Claim 4.2 $b+\beta\neq -2$.

Proof of Claim 4.2. Assume for a contradiction that $b+\beta=-2$. Then $K_1(m_1)=S^2(-2,\ \beta_1/\alpha_1,\ \beta_2/\alpha_2,\ 1)=S^2(-1,\ \beta_1/\alpha_1,\ \beta_2/\alpha_2)$. Hence $|H_1(K_1(m_1))|=|-\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1|$, which coincides with $pq+l^2=\alpha_1\alpha_2+l^2$. Since $\alpha_1\alpha_2+l^2>\alpha_1\alpha_2$, we have $|-\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1|>\alpha_1\alpha_2$. This then implies $\beta_1/\alpha_1+\beta_2/\alpha_2>2$ or $\beta_1/\alpha_1+\beta_2/\alpha_2<0$. Either case cannot happen, because $0<\beta_i/\alpha_i<1$. Thus $b+\beta\neq -2$. □(Claim 4.2)

Claim 4.3 *If* $b+\beta=-1$, $\beta_1/\alpha_1+\beta_2/\alpha_2>1$.

Proof of Claim 4.3. If $b+\beta=-1$, then $K_1(m_1)=S^2(-1,\beta_1/\alpha_1,\beta_2/\alpha_2,1)=S^2(\beta_1/\alpha_1,\beta_2/\alpha_2)$. Thus $|H_1(K_1(m_1))|=\alpha_1\beta_2+\alpha_2\beta_1$, which coincides with $pq+l^2=\alpha_1\alpha_2+l^2$. Since $\alpha_1\alpha_2+l^2>\alpha_1\alpha_2$, we have $\alpha_1\beta_2+\alpha_2\beta_1>\alpha_1\alpha_2$. This shows $\beta_1/\alpha_1+\beta_2/\alpha_2>1$. □(Claim 4.3)

Claims 4.2 and 4.3, together with the argument in the proof of Theorem 1.6 prove that K_n is an L-space knot for any $n\geq -1$.

Now let us prove that K_n is an L-space knot for all integers n under the assumption $l^2\geq 2pq$.

Claim 4.4 *If* $l^2\geq 2pq$*, then* $b+\beta\neq -1$.

Proof of Claim 4.4. Assume that $l^2 \geq 2pq$, and suppose for a contradiction $b + \beta = -1$. Then $K_{-1}(m_{-1}) = S^2(-1, \beta_1/\alpha_1, \beta_2/\alpha_2, -1) = S^2(-2, \beta_1/\alpha_1, \beta_2/\alpha_2)$, and $|H_1(K_{-1}(m_{-1}))| = |-2\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1|$, which coincides with $|pq - l^2|$. The assumption $l^2 \geq 2pq = 2\alpha_1\alpha_2$ implies that $|pq - l^2| = l^2 - pq = l^2 - \alpha_1\alpha_2 \geq \alpha_1\alpha_2$. Hence $|-2\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1| = |pq-l^2| \geq \alpha_1\alpha_2$. Thus we have $\beta_1/\alpha_1+\beta_2/\alpha_2 \geq 3$ or $\beta_1/\alpha_1 + \beta_2/\alpha_2 \leq 1$. The former case cannot happen because $0 < \beta_i/\alpha_i < 1$, and the latter case contradicts Claim 4.3 which asserts $\beta_1/\alpha_1 + \beta_2/\alpha_2 > 1$. Hence $b + \beta \neq -1$. □(Claim 4.4)

Claim 4.5 *If $l^2 \geq 2pq$, then $b + \beta \neq 0$.*

Proof of Claim 4.5. Suppose for a contradiction that $b + \beta = 0$. Then $K_{-1}(m_{-1}) = S^2(0, \beta_1/\alpha_1, \beta_2/\alpha_2, -1) = S^2(-1, \beta_1/\alpha_1, \beta_2/\alpha_2)$, and $|H_1(K_{-1}(m_{-1}))| = |-\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1|$, which coincides with $|pq - l^2|$. Since $l^2 \geq 2pq = 2\alpha_1\alpha_2$, $|pq - l^2| = l^2-pq = l^2-\alpha_1\alpha_2 \geq \alpha_1\alpha_2$. Thus we have $|-\alpha_1\alpha_2+\alpha_1\beta_2+\alpha_2\beta_1| = |pq-l^2| \geq \alpha_1\alpha_2$. This then implies $\beta_1/\alpha_1 + \beta_2/\alpha_2 \geq 2$ or $\beta_1/\alpha_1 + \beta_2/\alpha_2 \leq 0$. Either case cannot happen, because $0 < \beta_i/\alpha_i < 1$. Thus $b + \beta \neq 0$. □(Claim 4.5)

Under the assumption $l^2 \geq 2pq$, Claims 4.2, 4.4 and 4.5 imply that $b + \beta \leq -3$ or $b+\beta \geq 1$. Then the proof of Theorem 1.6 enables us to conclude that K_n is an L-space knot for all integers n. □(Theorem 1.7)

Example 4.6 Let K be a torus knot $T_{3,2}$ and c an unknotted circle depicted in Figure 4.2; the linking number between c and $T_{3,2}$ is 5. Then c coincides with $c^+_{3,2}$ in Section 5, and it is a seiferter for $(T_{3,2}, 6)$. Let K_n be a knot obtained from $T_{3,2}$ by n–twist along c. Since $5^2 > 2 \cdot 3 \cdot 2 = 12$, following Theorem 1.7 K_n is an L-space knot for all integers n.

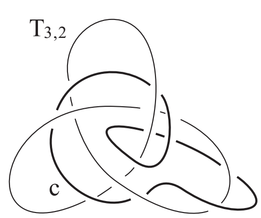

Figure 4.2: c is a seiferter for $(T_{3,2}, 6)$.

Example 4.7 below gives an example of a seiferter for (K, m), where K is a cable of a torus knot and $K(m)$ is a connected sum of two lens spaces.

Example 4.7 Let k be a Berge knot $Spor\mathbf{a}[p]$ ($p > 1$). Then $k(22p^2+9p+1)$ is a lens space and [11, Proposition 8.1 and Table 9] shows that $(k, 22p^2+9p+1)$ has a seiferter c such that the linking number between c and k is $4p+1$ and (-1)–twist along c converts $(k, 22p^2+9p+1)$ into $(C_{6p+1,p}(T_{3,2}), p(6p+1))$. Since $p > 1$, $C_{6p+1,p}(T_{3,2})$ is a nontrivial cable of $T_{3,2}$. Thus c is a seiferter for $(C_{6p+1,p}(T_{3,2}), p(6p+1))$. Let K_n be a knot obtained from $C_{6p+1,p}(T_{3,2})$ by n–twist along c so that $K_1 = k$. Since $(4p+1)^2 \geq 2(6p+1)p$, Theorem 1.7 shows that K_n is an L-space knot for all integers n.

Finally we show that K_n is hyperbolic if $|n| > 3$. As shown in [11, Figure 41], K_n admits a Seifert surgery yielding a small Seifert space which is not a lens space, so we see that c becomes a degenerate fiber in $C_{6p+1,p}(T_{3,2})(p(6p+1))$ [13, Lemma 5.6(1)]. Hence Corollary 3.21(3) in [13] shows that the link $C_{6p+1,p}(T_{3,2}) \cup c$ is hyperbolic. Now the result follows from [13, Proposition 5.11(3)].

We close this section with the following observation, which shows the non-uniqueness of degenerate Seifert fibration of a connected sum of two lens spaces.

Let c be a seiferter for $(T_{p,q}, pq)$ which becomes a degenerate fiber in $T_{p,q}(pq)$. As the simplest example of such a seiferter c, take a meridian c_μ of $T_{p,q}$. Then c_μ is isotopic to the core of the filled solid torus (i.e. the dual knot of $T_{p,q}$) in $T_{p,q}(pq)$, which is a degenerate fiber. Hence c_μ is a seiferter for $(T_{p,q}, pq)$ which becomes a degenerate fiber in $T_{p,q}(pq)$, and $T_{p,q} - \mathrm{int}N(c_\mu)$ is homeomorphic to $S^3 - \mathrm{int}N(T_{p,q})$. However, in general, $T_{p,q}(pq) - \mathrm{int}N(c)$ is not necessarily homeomorphic to $S^3 - \mathrm{int}N(T_{p,q})$.

Example 4.8 Let us take an unknotted circle c as in Figure 4.3. Then c is a seiferter for $(T_{5,3}, 15)$ which becomes a degenerate fiber in $T_{5,3}(15)$, but $T_{5,3}(15) - \mathrm{int}N(c)$ is not homeomorphic to $S^3 - \mathrm{int}N(T_{5,3})$.

Proof of Example 4.8. As shown in Figure 4.3, $T_{5,3}(15)$ is the the two-fold branched cover of S^3 branched along L' and c is the preimage of an arc τ. Hence $T_{5,3}(15) - \mathrm{int}N(c)$ is a Seifert fiber space $D^2(2/3, -2/5)$. Since $|H_1(S^2(2/3, -2/5, x))| = |4 + 15x|$ cannot be 1 for any integer x, the Seifert fiber space $T_{5,3}(15) - \mathrm{int}N(c)$ cannot be embedded in S^3, and hence it is not homeomorphic to $S^3 - \mathrm{int}N(T_{5,3})$. (Note that c coincides with $c^-_{5,3}$ in Section 5.) □(Example 4.8)

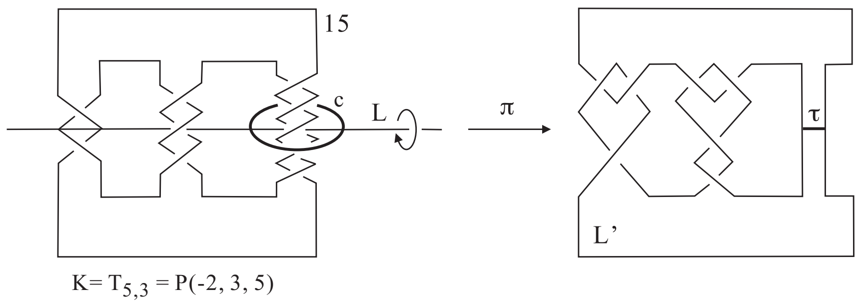

Figure 4.3: $T_{5,3}(15)$ is the the two-fold branched cover of S^3 branched along L'.

5 L-space twisted torus knots

Each torus knot has obviously an unknotted circle c which satisfies the desired property in Question 1.1.

Example 5.1 Embed a torus knot $T_{p,q}$ into a genus one Heegaard surface of S^3. Then cores of the Heegaard splitting s_p and s_q are seiferters for $(T_{p,q}, m)$ for all integers m. We call them *basic seiferters* for $T_{p,q}$; see Figure 5.1. An n–twist along s_p (resp. s_q) converts $T_{p,q}$ into a torus knot $T_{p+nq,q}$ (resp. $T_{p,q+np}$), and hence n–twist along a basic seiferter yields an L-space knot for all n.

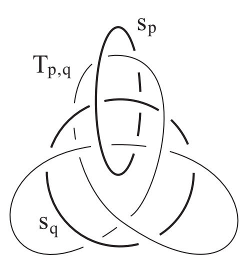

Figure 5.1: s_p and s_q are basic seiferters for $(T_{p,q}, m)$.

Twistings along a basic seiferter keep the property of being L-space knots, but produce only torus knots. In the following, we will give another circle c such that twistings $T_{p,q}$ along c produce an infinite family of hyperbolic L-space knots.

Definition 5.2 (**twisted torus knot [10]**) Let Σ be a genus one Heegaard surface of S^3. Let $T_{p,q}$ $(p > q \geq 2)$ be a (p, q)–torus knot which lies on Σ. Choose an unknotted circle $c \subset S^3 - T_{p,q}$ so that it bounds a disk D such that $D \cap \Sigma$ is a single arc intersecting $T_{p,q}$ in r $(2 \leq r \leq p+q)$ points in the same direction. A *twisted torus knot* $K(p,q;r,n)$ is a knot obtained from $T_{p,q}$ by adding n full twists along c.

Remark 5.3 Twisting $T_{p,q}$ along the basic seiferter s_p (resp. s_q) n–times, we obtain the twisted torus knot $K(p,q;q,n)$ (resp. $K(p,q;p,n)$), which is a torus knot $T_{p+nq,q}$ (resp. $T_{p,q+np}$), and hence an L-space knot.

In [53] Vafaee studies twisted torus knots from a viewpoint of knot Floer homology and showed that twisted torus knots $K(p,kp\pm 1;r,n)$, where $p \geq 2, k \geq 1, n > 0$ and $0 < r < p$ is an L-space knot if and only if either $r = p-1$ or $r \in \{2, p-2\}$ and $n = 1$. We will give yet more twisted torus knots which are L-space knots by combining seiferter technology and Theorem 1.7.

Proof of Theorem 1.8. In the following, let Σ be a genus one Heegaard surface of S^3, which bounds solid tori V_1 and V_2.

• $K(p,q;p+q,n)$ $(p > q \geq 2)$. Given any torus knot $T_{p,q}$ $(p > q \geq 2)$ on Σ, let us take an unknotted circle $c^+_{p.q}$ in $S^3 - T_{p,q}$ as depicted in Figure 5.2(i); the linking number between $c^+_{p,q}$ and $T_{p,q}$ is $p+q$.

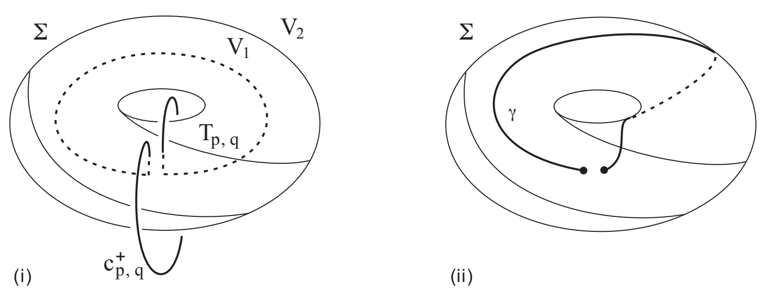

Figure 5.2: $c^+_{p,q}$ is a seiferter for $(T_{p,q}, pq)$.

Let V be a solid torus $S^3 - \mathrm{int}N(c^+_{p,q})$, which contains $T_{p,q}$ in its interior. Lemma 9.1 in [36] shows that $V(K;pq) = T_{p,q}(pq) - \mathrm{int}N(c^+_{p,q})$ is a Seifert fiber space over the disk with two exceptional fibers of indices p,q, and a meridian of $N(c^+_{p,q})$ coincides with a regular fiber on $\partial V(K;pq)$. Hence $c^+_{p,q}$ is a degenerate fiber in $T_{p,q}(pq)$, and thus it is a seiferter for $(T_{p,q}, pq)$. Let D be a disk bounded by $c^+_{p,q}$. Since the arc $c^+_{p,q} \cap V_i$ is isotoped in V_i to an arc $\gamma \subset \Sigma$ depicted in Figure 5.2(ii) leaving its endpoints fixed, the disk D can be isotoped so that $D \cap \Sigma = \gamma$, which intersects $T_{p,q}$ in $p+q$ points in the same direction. Thus n–twist along $c^+_{p,q}$ converts $T_{p,q}$

into the twisted torus knot $K(p,q;p+q,n)$. Since $c^+_{p,q}$ is a seiferter for $(T_{p,q},pq)$ and $(p+q)^2 = p^2+q^2+2pq > 2pq$, we can apply Theorem 1.7 to conclude that $T(p,q,p+q,n)$ is an L-space knots for all integers n.

We show that $T(p,q,p+q,n)$ is hyperbolic if $|n| > 3$. By linking number consideration, we see that $c^+_{p,q}$ is not a basic seiferter. Then Corollary 3.21(3) in [13] ([36, Claim 9.2]) shows that $T_{p,q} \cup c^+_{p,q}$ is a hyperbolic link. Thus [13, Proposition 5.11(2)] that $K(p,q;p+q,n)$ is a hyperbolic knot if $|n| > 3$.

• $K(p,q;p-q,n)$ $(p > q \geq 2)$. Suppose that $p-q \neq 1$. Then let us take $c^-_{p,q}$ as in Figure 5.3(i) instead of $c^+_{p,q}$; the linking number between $c^-_{p,q}$ and $T_{p,q}$ is $p-q$. It follows from [13, Remark 4.7] that $c^-_{p,q}$ is also a seiferter for $(T_{p,q},pq)$ and the link $T_{p,q} \cup c^-_{p,q}$ is hyperbolic. Note that if $p-q=1$, then $c^-_{p,q}$ is a meridian of $T_{p,q}$. As above we see that each arc $c^-_{p,q} \cap V_i$ is isotoped in V_i to an arc $\gamma \subset \Sigma$ depicted in Figure 5.3(ii) leaving its endpoints fixed. So a disk D bounded by $c^-_{p,q}$ can be isotoped so that $D \cap \Sigma = \gamma$, which intersects $T_{p,q}$ in $p-q$ points in the same direction. Thus n–twist along $c^-_{p,q}$ converts $T_{p,q}$ into the twisted torus knot $K(p,q;p-q,n)$. Since $c^-_{p,q}$ is a seiferter for $(T_{p,q},pq)$, Theorem 1.7 shows that $T(p,q,p-q,n)$ is an L-space knot for any $n \geq -1$. Following [13, Proposition 5.11(2)] $T(p,q,p-q,n)$ is a hyperbolic knot if $|n| > 3$.

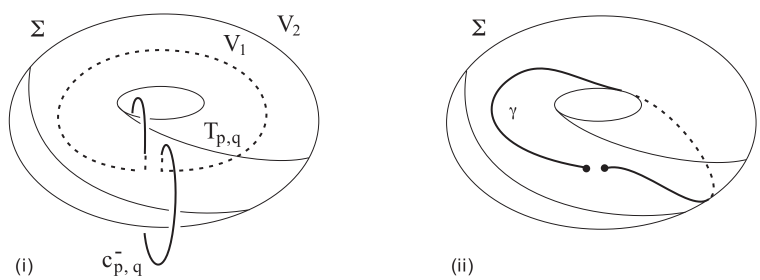

Figure 5.3: $c^-_{p,q}$ is a seiferter for $(T_{p,q},pq)$.

• $K(3p+1,2p+1;4p+1,n)$ $(p>0)$. Let k be a torus knot $T_{p,2p+1}$ on a genus two Heegaard surface, with unknotted circles α and c as shown in Figure 5.4. Applying 1–twist along α, we obtain a torus knot $T_{3p+1,2p+1}$. We continue to use the same symbol c to denote the image of c after 1–twist along α; the linking number between c and $T_{3p+1,2p+1}$ is $4p+1$. Note that 1–twist along c converts $T_{3p+1,2p+1}$ into a Berge knot *Spor*$\mathbf{b}[p]$ as shown in [11, Subsection 8.2]. Following [11, Lemma 8.4] c is a seiferter for a lens space surgery $(\textit{Spor}\mathbf{b}[p], 22p^2+13p+2) = (\textit{Spor}\mathbf{b}[p], (3p+1)(2p+1)+(4p+1)^2)$. Thus c is also a seiferter for $(T_{3p+1,2p+1},(3p+1)(2p+1))$. Let

D be a disk bounded by c. Then $T_{3p+1,2p+1} \cup D$ can be isotoped so that $T_{3p+1,2p+1}$ lies on Σ, $D \cap \Sigma$ consists of a single arc, which intersects $T_{3p+1,2p+1}$ in $4p+1$ points in the same direction. Thus n–twist along c converts $T_{3p+1,2p+1}$ into a twisted torus knot $K(3p+1, 2p+1; 4p+1, n)$. Since c is a seiferter for $(T_{3p+1,2p+1}, (3p+1)(2p+1))$ and $(4p+1)^2 > 2(3p+1)(2p+1)$, Theorem 1.7 shows that $K(3p+1, 2p+1; 4p+1, n)$ is an L-space knot for all integers n. Let us observe that $K(3p+1, 2p+1; 4p+1, n)$ is a hyperbolic knot if $|n| > 3$. Figure 44 in [11] shows that n–twist converts $(T_{3p+1,2p+1}, (3p+1)(2p+1))$ into a Seifert surgery which is not a lens space surgery if $|n| \geq 2$. Hence c becomes a degenerate fiber in $T_{3p+1,2p+1}((3p+1)(2p+1))$ [13, Lemma 5.6(1)], and Corollary 3.21(3) in [13] shows that the link $T_{3p+1,2p+1} \cup c$ is hyperbolic. Now the result follows from [13, Proposition 5.11(2)].

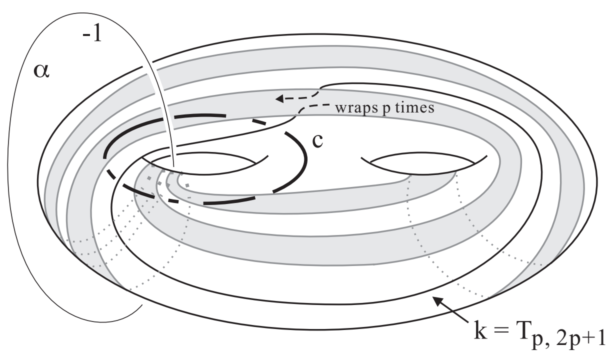

Figure 5.4: A surgery description of $T_{3p+1,2p+1}$ and a seiferter c

• $K(3p+2, 2p+1; 4p+3, n)$ $(p > 0)$. As above, we follow the argument in [11, Subsection 8.3], but we need to take the mirror image at the end. Take a torus knot $k = T_{-p-1,2p+1}$ on a genus two Heegaard surface of S^3, unknotted circles α' and c' as shown in Figure 5.5. Then (-1)–twist along α' converts $T_{-p-1,2p+1}$ into $T_{-3p-2,2p+1}$. As above we denote the image of c' after (-1)–twist along α' by the same symbol c'; the linking number between c' and $T_{-3p-2,2p+1}$ is $4p+3$. Note that (-1)–twist along c' converts $T_{-3p-2,2p+1}$ into a Berge knot $Spor\mathbf{c}[p]$ as shown in [11, Subsection 8.3]. Then Lemma 8.6 in [11] shows that c' is a seiferter for a lens space surgery $(Spor\mathbf{c}[p], -22p^2 - 31p - 11) = (Spor\mathbf{c}[p], (-3p-2)(2p+1) - (4p+3)^2)$. Thus c' is also a seiferter for $(T_{-3p-2,2p+1}, (-3p-2)(2p+1))$. Let D' be a disk bounded by c'. Then $T_{-3p-2,2p+1} \cup D'$ can be isotoped so that $T_{-3p-2,2p+1}$ lies on Σ, $D' \cap \Sigma$ consists of a single arc, which intersects $T_{-3p-2,2p+1}$ in $4p+3$ points in the same direction. Now taking the mirror image of $T_{-3p-2,2p+1} \cup D'$, we obtain $T_{3p+2,2p+1} \cup D$ with $\partial D = c$; $D \cap \Sigma$ consists of a single arc, and D intersects $T_{3p+2,2p+1}$ in $4p+3$ points in the same direction. Then c is a seiferter for $(T_{3p+2,2p+1}, (3p+2)(2p+1))$. Since $(4p+3)^2 > 2(3p+2)(2p+1)$, Theorem 1.7 shows that $K(3p+2, 2p+1; 4p+3, n)$ is an L-space knot for all integers n. Let us show that $K(3p+2, 2p+1; 4p+3, n)$ is hyperbolic if $|n| > 3$. Figure 47 in [11], together with [13, Lemma 5.6(1)], shows that c' becomes a degenerate fiber in $T_{-3p-2,2p+1}((-3p-2)(2p+1))$, and so c becomes a degenerate fiber in $T_{3p+2,2p+1}((3p+2)(2p+1))$. Apply the same argument as above to obtain the desired result.

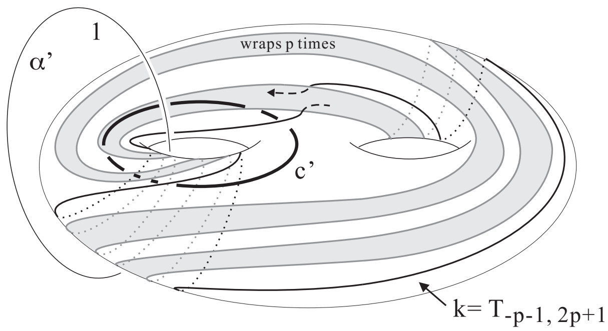

Figure 5.5: A surgery description of $T_{-3p-2,2p+1}$ and a seiferter c'

• $K(2p+3, 2p+1; 2p+2, n)$ $(p > 0)$. We follow the argument in [11, Section 6]; as above we need to take the mirror image at the end. Take a torus knot $k = T_{-3p-2,3}$ on a genus two Heegaard surface of S^3, unknotted circles α' and c' as in Figure 5.6(i). Then (-2)–twist along α' converts the torus knot $T_{-3p-2,3}$ into a Berge knot VI[p]. Lemma 6.1 in [11] shows that c', the image of c' after the (-2)–twist along α', is a seiferter for a lens space surgery (VI[p], $-8p^2-16p-7$); the linking number between c' and VI[p] is $2p+2$. Now we show that 1–twist along c' (after (-2)–twist along α') converts (VI[p], $-8p^2-16p-7$) into $(T_{-2p-1,2p+3}, (-2p-1)(2p+3))$. Note that c' remains a seiferter for $(T_{-2p-1,2p+3}, (-2p-1)(2p+3))$. Since the linking number between c' and VI[p] is $2p+2$, the surgery slope $-8p^2-16p-7$ becomes $-8p^2-16p-7+(2p+2)^2 = (-2p-1)(2p+3)$. Let us observe that the knot obtained from VI[p] by 1–twist along c', which has a surgery description given by Figure 5.6(i), is $T_{-2p-1,2p+3}$. The surgeries described in Figure 5.6(i) can be realized by the following two successive twistings: 1–twist along an annulus cobounded by c' and α' (cf.[13, Definition 2.32]), and (-1)–twist along α'. The annulus twist converts $k = T_{-3p-2,3}$ into $k' = T_{-2p-1,2}$ as shown in Figure 5.6(ii). Then (-1)–twist along α' changes $k' = T_{-2p-1,2}$ into $T_{-2p-1,2p+3}$, which lies on the genus one Heegaard surface Σ. Let D' be a disk bounded by c'. Then D' can be slightly isotoped so that $D' \cap \Sigma$ consists of a single arc, which intersects $T_{-2p-1,2p+3}$ in $2p+2$ points in the same direction; see Figure 5.6(ii). Now taking the mirror image of $T_{-2p-1,2p+3} \cup D'$, we obtain $T_{2p+1,2p+3} \cup D$ with $\partial D = c$; $D \cap \Sigma$ consists of a single arc, which intersects $T_{2p+1,2p+3}$ in $2p+2$ points in the same direction. Then c is a seiferter for $(T_{2p+1,2p+3}, (2p+1)(2p+3)) = (T_{2p+3,2p+1}, (2p+3)(2p+1))$. Theorem 1.7 shows that $K(2p+3, 2p+1; 2p+2, n)$ is an L-space knot for any integer $n \geq -1$. The hyperbolicity of knots $K(2p+3, 2p+1; 2p+2, n)$ for $|n| > 3$ follows from the same argument as above, in which we refer to Figure 33 instead of Figure 47 in [11].

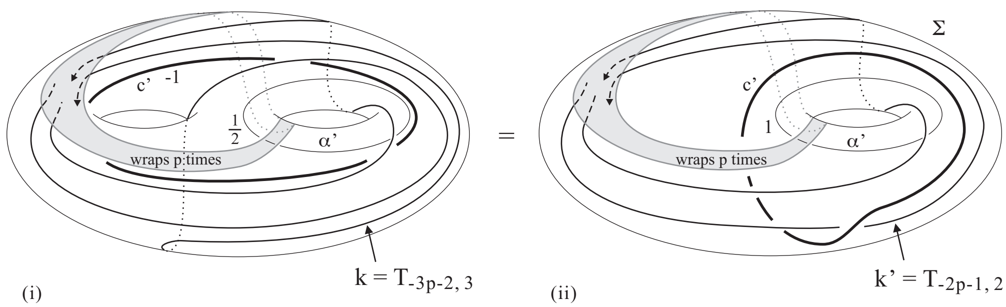

Figure 5.6: Surgery descriptions of $T_{-2p-1,2p+3}$ and a seiferter c'

□(Theorem 1.8)

Proof of Corollary 1.9. Given any torus knot $T_{p,q}$ $(p > q \geq 2)$, let us take an unknotted circle $c = c^+_{p.q}$ in $S^3 - T_{p,q}$ (Figure 5.2(i)). Then as shown in the proof of Theorem 1.8, n–twist along c converts $T_{p,q}$ into the twisted torus knot $K(p, q; p+q, n)$, which is an L-space knot for all integers n and hyperbolic if $|n| > 3$.

The last assertion of Corollary 1.9 follows from Claim 5.4 below. Thus the unknotted circle c satisfies the required property in Corollary 1.9. □(Corollary 1.9)

Claim 5.4 $\{K(p, q; p+q, n)\}_{|n|>3}$ *is a set of mutually distinct hyperbolic knots.*

Proof of Claim 5.4. Recall that $c^+_{p,q}$ is a seiferter for $(T_{p,q}, pq)$ and the linking number between $c^+_{p,q}$ and $T_{p,q}$ is $p+q$. Thus n–twist along $c^+_{p,q}$ changes $(T_{p,q}, pq)$ to a Seifert surgery $(K(p, q; p+q, n), pq + n(p+q)^2)$. Note that $K(p, q; p+q, n)(pq + n(p+q)^2)$ is a Seifert fiber space over S^2 with at most three exceptional fibers of indices p, q and $|n|$, see the proof of Theorem 1.8.

Assume that $K(p, q; p+q, n)$ is isotopic to $K(p, q; p+q, n')$ for some integers n, n' with $|n|, |n'| > 3$. Then $pq+n(p+q)^2$–, and $pq+n'(p+q)^2$–surgeries on the hyperbolic knot $K(p, q; p+q, n)$ yield Seifert fiber spaces. Hence $|pq+n(p+q)^2 - (pq+n'(p+q)^2)| =$

$|(n-n')(p+q)^2| \leq 8$ by [32, Theorem 1.2]. Since $p+q \geq 5$, we have $n = n'$. This completes a proof. (In the above argument, we can apply [1, Theorem 8.1] which gives the bound 10 instead of 8.) $\square$(Claim 5.4)

6 L-space twisted Berge knots

In this section we prove Theorem 1.11 using Theorem 1.7 and observations in [13, 11].

Berge [6] gave twelve infinite families of knots which admit lens space surgeries. These knots are referred to as *Berge knots* of types (I)–(XII) and conjectured to comprise all knots with lens space surgeries. Recall that a Berge knot of type (I) is a torus knot and that of (II) is a cable of a torus knot, henceforth we consider Berge knots of types (III)–(XII).

• Berge knots of types (III)–(VI).

Suppose that K is a Berge knot of type (III), (IV), (V) or (VI). Then we have an unknotted solid torus V containing K in its interior such that $V(K;m)$ is a solid torus [6, 11], and hence the core c of the solid torus $W = S^3 - \mathrm{int}V$ is a seiferter for (K,m) and (K_n, m_n) is also a lens space. If $K_n(m_n)$ is not an L-space, then it is $S^2 \times S^1$ and $(K_n, m_n) = (O, 0)$ ([17, Theorem 8.1]). Now let us exclude this possibility. First we note that $V(K_n, m_n) \cong V(K;m)$ for all integers n and $H_1(V(K_n;m_n)) \cong \mathbb{Z} \oplus \mathbb{Z}_{(m_n,\omega)}$ [20, Lemma 3.3], where ω is the winding number of K in V, i.e. the linking number between K_n and c. Since $V(K_n;m_n) \cong S^1 \times D^2$, K_n is a 0 or 1–bridge braid in V [18], hence $\omega \geq 2$. This then implies that $m_n \neq 0$. Hence (K_n, m_n) is an L-space knot for all integers n.

• Berge knots of types (VII), (VIII).

Let g_1 and g_2 be simple closed curves embedded in a genus two Heegaard surface F of S^3 and c an unknot in S^3 as in Figure 6.1.

Take a regular neighborhood $N(g_1 \cup g_2)$ of $g_1 \cup g_2$ in F, which is a once punctured torus. Then the curve $\partial N(g_1 \cup g_2)$ becomes a trefoil knot after (-1)–twist along c, and the figure-eight knot after 1–twist along c. Let k be a knot in $N(g_1 \cup g_2)$ representing $a[g_1] + b[g_2] \in H_1(N(g_1 \cup g_2))$, where a and b are coprime integers. The we see that k is a torus knot $T_{a+b,-a}$. The Berge knot K of type (VII) (resp. (VIII)) is obtained from $T_{a+b,-a}$ by (-1)–twist (resp. 1–twist) along c. As shown in

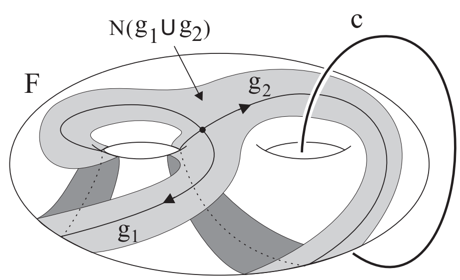

Figure 6.1: A regular neighborhood $N(g_1 \cup g_2)$ of $g_1 \cup g_2$ in F and an unknotted circle c

[13, Lemma 4.6], $T_{a+b,-a} \cup c$ is isotopic to $T_{a+b,-a} \cup c^+_{a+b,-a}$ and Berge knot of type (VII) is $K(a+b,-a;|b|,-1)$, that of type (VIII) is $K(a+b,-a;|b|,1)$; see the proof of Theorem 1.8. (Here we extend the notation $K(p,q;r,n)$ for twisted torus knots in an obvious fashion to include the case where p,q are possibly negative integers.) We assume $|a|,|b| \geq 2$, for otherwise $K(a+b,-a,|b|,\pm 1)$ is a torus knot. Furthermore, if $|a+b| = 1$, then $T_{a+b,-a} \cup c = T_{\pm 1,-a} \cup c$ is a torus link $T_{2,2b}$ or $T_{2,-2b}$, and $K(a+b,-a;|b|,\pm 1)$ is a torus knot, so we assume $|a+b| > 1$. Let K_n be a knot obtained from the Berge knot K by n–twist along c, i.e. $K_n = K(a+b,-a;|b|,n+\varepsilon)$; $\varepsilon = -1$ if K is of type (VII), $\varepsilon = 1$ if K is of type (VIII). If $a(a+b) < 0$ (i.e. $-a(a+b) > 0$), then by Theorem 1.8 K_n is an L-space knot for any integer n. If $a(a+b) > 0$ (i.e. $-a(a+b) < 0$), Theorem 1.8 shows that the mirror image $K(a+b,a;|b|,-n-\varepsilon)$ of K_n is an L-space knot if $-n-\varepsilon \geq -1$, i.e. $n \leq 1-\varepsilon$. Hence K_n is an L-space knot for any integer $n \leq 1-\varepsilon$.

• Berge knots of types (IX)–(XII).

These knots are often called *sporadic* knots and we denote them by $Spor\mathbf{a}[p]$, $Spor\mathbf{b}[p]$, $Spor\mathbf{c}[p]$ and $Spor\mathbf{d}[p]$ $(p \geq 0)$, respectively. It is easy to see that $Spor\mathbf{a}[0]$ and $Spor\mathbf{b}[0]$ are trivial knots, $Spor\mathbf{c}[0] = T_{-3,4}$ and $Spor\mathbf{c}[0] = T_{-5,3}$. Thus we may assume $p > 0$ for $Spor\chi[p]$ $(\chi = \mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d})$. Furthermore, we observe that $Spor\mathbf{a}[1]$ is obtained from $T_{3,2}$ by 1–twist along the seiferter $c = c^+_{3,2}$; see Figure 4.2. Hence following Example 4.6 a knot K_n obtained from $Spor\mathbf{a}[1]$ by n–twist along c is an L-space knot for any integer n. Thus we may assume $p > 1$ for $Spor\mathbf{a}[p]$.

As shown in Example 4.7 the lens space surgery $(Spor\mathbf{a}[p], 22p^2+9p+1)$ is obtained from $(C_{6p+1,p}(T_{3,2}), p(6p+1))$ by 1–twist along the seiferter c, and n–twist along c converts $C_{6p+1,p}(T_{3,2})$ into an L-space knot for all integers n. Hence n–twist changes $Spor\mathbf{a}[p]$ to an L-space knots for all integers n.

The proof of Theorem 1.8 shows that the lens space surgery $(Spor\mathbf{b}[p], 22p^2+13p+2)$ is obtained from $(T_{3p+1,2p+1}, (3p+1)(2p+1))$ by 1–twist along the seiferter c, hence K_n obtained from $Spor\mathbf{b}[p]$ by n–twist along c is $K(3p+1, 2p+1; 4p+1, n+1)$. By Theorem 1.8 K_n is an L-space knot for all integers n. Similarly $(Spor\mathbf{c}[p], -22p^2-31p-11)$ is obtained from $(T_{-3p-2,2p+1}, (-3p-2)(2p+1))$ by (-1)–twist along c', and K_n obtained from $Spor\mathbf{c}[p]$ by n–twist along c' is $K(-3p-2, 2p+1; 4p+4, n-1)$, which is the mirror image of $K(3p+2, 2p+1; 4p+4, -n+1)$. Theorem 1.8 shows that $K(3p+2, 2p+1; 4p+4, -n+1)$ is an L-space knot for any integer n, and thus K_n is an L-space knot for all integers n.

Finally, let us consider a Berge knot $Spor\mathbf{d}[p]$ $(p \geq 0)$. Proposition 8.8 in [11] shows that the lens space surgery $(Spor\mathbf{d}[p], -22p^2-35p-14)$ has a seiferter c' such that the linking number between c' and $Spor\mathbf{d}[p]$ is $4p+3$ and 1–twist along c' converts $(Spor\mathbf{d}[p], -22p^2-35p-14)$ into $(C_{-6p-5,p+1}(T_{-3,2}), (-6p-5)(p+1))$ for which c' is a seiferter. Let K_n be a knot obtained from $Spor\mathbf{d}[p]$ by n–twist along c', i.e. obtained from $C_{-6p-5,p+1}(T_{-3,2})$ by $(n-1)$–twist along c'. Now we take the mirror image of $C_{-6p-5,p+1}(T_{-3,2}) \cup c'$ to obtain a link $C_{6p+5,p+1}(T_{3,2}) \cup c$. Then c is a seiferter for $(C_{6p+5,p+1}(T_{3,2}), (6p+5)(p+1))$ and K_n is the mirror image of the knot obtained from $C_{6p+5,p+1}(T_{3,2})$ by $(-n)$–twist along c. Since $(4p+3)^2 \geq 2(6p+5)(p+1)$, Theorem 1.7 shows that K_n is an L-space knot for all integers n.

Let us show that K_n is a hyperbolic knot except for at most four integers n. Following [13, Theorem 5.10] it is sufficient to observe that $K \cup c$ is a hyperbolic link. Suppose that K is a Berge knot of type (III), (IV), (V) or (VI). Then as mentioned above, $V(K; m)$ is a solid torus, where $V = S^3 - \mathrm{int}N(c)$. By [5, Theorem 3.2] $V - \mathrm{int}N(K)$ is atoroidal. If $V - \mathrm{int}N(K)$ is not hyperbolic, then it is Seifert fibered and K is a torus knot; see [13, Lemma 3.3]. This contradicts the assumption. Hence $K \cup c$ is a hyperbolic link.

If K is of type (VII) or (VIII), then $K \cup c \cong T_{a+b,-a} \cup c^+_{a+b,-a}$ is a hyperbolic link; see the proof of Theorem 1.8.

Assume that K is of type (IX), i.e. $K = Spor\mathbf{a}[p]$. Then as shown in the proof of Example 4.7 $K \cup c$ is a hyperbolic link. In the case where K is of type (X) or (XI), i.e. $K = Spor\mathbf{b}[p]$ or $Spor\mathbf{c}[p]$, then it follows from the proof of Theorem 1.8 that $K \cup c$ is a hyperbolic link. The argument in the proof of Example 4.7 shows $K \cup c$ is a hyperbolic link for type (XII) Berge knot $K = Spor\mathbf{d}[p]$; we refer to Figure 53 instead of Figure 41.

This completes a proof of Theorem 1.11. □(Theorem 1.11)

7 L-space twisted unknots

In [13] we introduced "m–move" to find seiferters for a given Seifert surgery. In particular, m–move is effectively used in [13, Theorem 6.21] to show that (O, m) has infinitely many seiferters for each integer m. Among them there are infinitely many seiferters c such that $(m, 0)$–surgery on $O \cup c$ is an L-space; see Remark 7.3.

Let us take a trivial knot $c_{m,p}$ in $S^3 - O$ as illustrated in Figure 7.1, where p is an odd integer with $|p| \geq 3$.

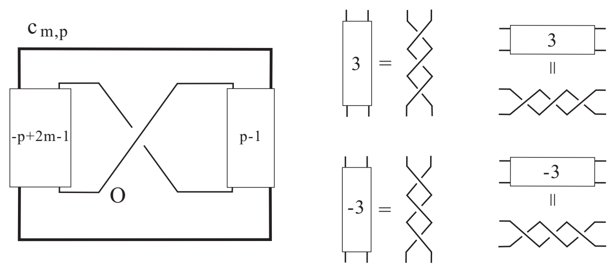

Figure 7.1: $O \cup c_{m,p}$; a vertical (resp. horizontal) box with integer n denotes a vertical (resp. horizontal) stack of n crossings.

Then as shown in [13, Theorem 6.21] $c_{m,p}$ is a seiferter for (O, m) such that $O \cup c_{m,p}$ is a hyperbolic link in S^3 if $p \neq 2m \pm 1$. Denote by $K_{m,p,n}$ and $m_{p,n}$ the images of O and m after n–twist along $c_{m,p}$. Now we investigate $K_{m,p,n}(m_{p,n})$ using branched coverings and Montesinos trick [38, 39]. Figure 7.2 (b) shows that $K_{m,p,n}(m_{p,n})$ has an involution with axis L for any integer n. Taking the quotient by this involution, we obtain a 2–fold branched cover $\pi : K_{m,p,n}(m_{p,n}) \to S^3$ branched along L' which is the quotient of L; see Figure 7.2(c). As shown in Figure 7.2(d) L' can be isotoped to a Montesinos link $M(-n/(mn+1),\ (-p+1)/2p,\ (-p+2m+1)/(-2p+4m))$. Hence by [38] $K_{m,p,n}(m_{p,n})$, which is the 2–fold branched cover branched along the Montesinos link L', is a Seifert fiber space

$$S^2(\frac{-n}{mn+1}, \frac{-p+1}{2p}, \frac{p-2m-1}{2p-4m}).$$

The image $\pi(c_{m,p})$ is an arc τ whose ends lie in L'; see Figure 7.2(c) and (d). It follows from [12, Lemma 3.2] that $c_{m,p}$ is a seiferter for $(K_{m,p,n}, m_{p,n})$; in case of $n = 0$, $c_{m,p}$ is a seiferter for (O, m). In the following, the image of $c_{m,p}$ after n–twist along itself is denoted by the same symbol.

In what follows assume $m \leq 0$ and $p \geq 3$.

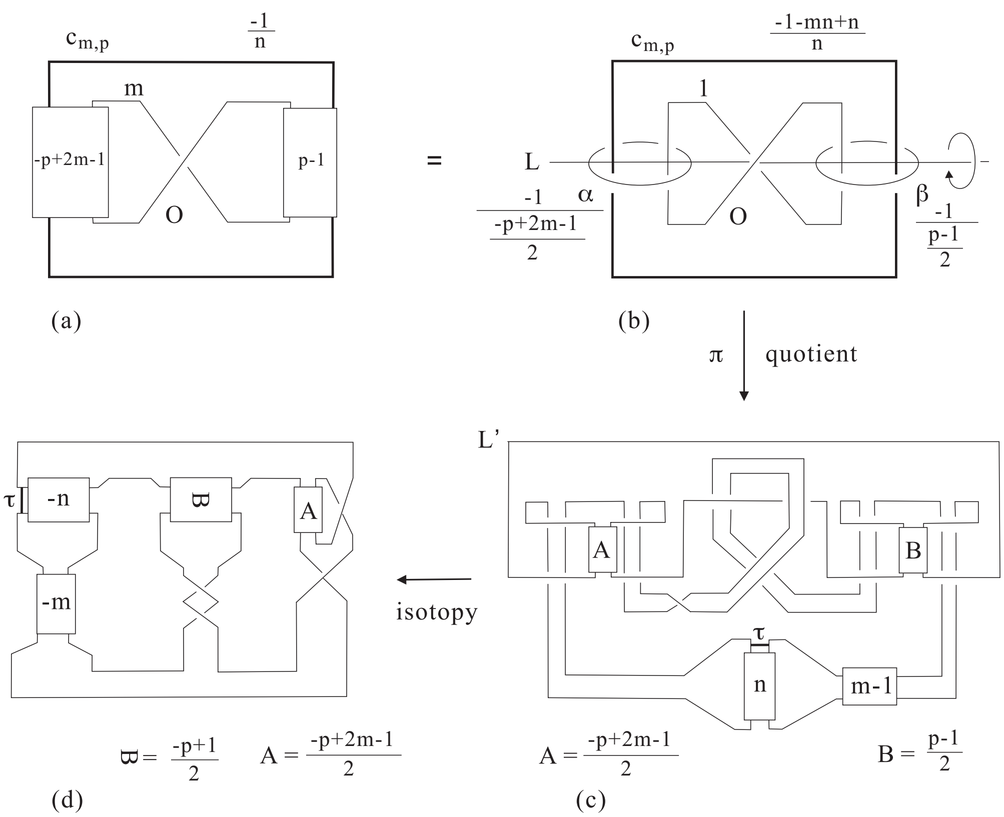

Figure 7.2: $K_{m,p,n}(m_{p,n})$ is the two-fold branched cover of S^3 branched along L'.

Proposition 7.1 *Assume that $m \le 0,\ p \ge 3$.*

(1) *$(K_{m,p,n}, m_{p,n})$ is an L-space surgery except when $(m,n) = (0,0)$. If $(m,n) = (0,0)$, then $(K_{m,p,n}, m_{p,n}) = (O,0)$ and $K_{m,p,n}(m_{p,n}) = O(0) \cong S^2 \times S^1$.*

(2) *$K_{m,p,n}$ is a nontrivial knot if $n \neq 0$.*

(3) *$\{K_{m,p,n}\}_{|n|>1}$ is a set of mutually distinct hyperbolic L-space knots.*

Proof of Proposition 7.1. We note here that the linking number between $c_{m,p}$ and O is $p - m$.

(1) Assume first that $m = 0$. Then $K_{m,p,n}(m_{p,n})$ is a lens space $S^2(-n,\ (-p+1)/2p,\ (p-1)/2p) = S^2(-n-1,\ (p+1)/2p,\ (p-1)/2p)$, which is $S^2 \times S^1$ if and only if $n = 0$ (Claim 2.5). Hence $K_{m,p,n}(m_{p,n})$ is an L-space except when $n = 0$.

Next assume $m = -1$. Then $K_{m,p,n}(m_{p,n}) = S^2(-n/(-n+1),\ (-p+1)/2p,\ (p+1)/(2p+4)) = S^2(n/(n-1),\ (-p+1)/2p,\ (p+1)/(2p+4))$. If $n = 0$ or 2, $K_{m,p,n}(m_{p,n})$ is a lens space, but it is not $S^2 \times S^1$, because $m_{p,n} = -1 + n(m-p)^2 = -1 + n(p+1)^2 \neq 0$. If $n = 1$, $K_{m,p,n}(m_{p,n})$ is a connected sum of two lens spaces, and thus an L-space. Suppose that $n \neq 0, 1, 2$. In case of $n < 0$, we have $0 < n/(n-1) < 1$ and $K_{m,p,n}(m_{p,n}) = S^2(n/(n-1),\ (-p+1)/2p,\ (p+1)/(2p+4)) = S^2(-1,\ n/(n-1),\ (p+1)/2p,\ (p+1)/(2p+4))$. Note that $(p+1)/2p + (p+1)/(2p+4) = 1/2 + 1/2p + 1/2 - 1/(2p+4) = 1 + 1/2p - 1/(2p+4)$. Since $p \ge 3$, we have $2p+4 > 2p > 0$, and hence $1/2p - 1/(2p+4) > 0$. It follows that $(p+1)/2p + (p+1)/(2p+4) = 1 + 1/2p - 1/(2p+4) > 1$. Then Lemma 2.3(2) shows that $K_{m,p,n}(m_{p,n})$ is an L-space. If $n > 2$, then $1 < n/(n-1) < 2$ and $K_{m,p,n}(m_{p,n}) = S^2(n/(n-1),\ (-p+1)/2p,\ (p+1)/(2p+4)) = S^2(1/(n-1),\ (p+1)/2p,\ (p+1)/(2p+4))$. Since $0 < 1/(n-1),\ (p+1)/2p,\ (p-2m-1)/(2p-4m) < 1$, $K_{m,p,n}(m_{p,n})$ is an L-space by Theorem 2.1(1).

Assume that $m = -2$. Then $K_{m,p,n}(m_{p,n}) = S^2(-n/(-2n+1),\ (-p+1)/2p,\ (p+3)/(2p+8)) = S^2(n/(2n-1),\ (-p+1)/2p,\ (p+3)/(2p+8))$. If $n = 0, 1$, then $K_{m,p,n}(m_{p,n})$ is a lens space, but it is not $S^2 \times S^1$, because $m_{p,n} = -2 + n(m-p)^2 = -2 + n(p+2)^2 \neq 0$. Otherwise, $0 < n/(2n-1) < 1$ and $K_{m,p,n}(m_{p,n}) = S^2(n/(2n-1),\ (-p+1)/2p,\ (p+3)/(2p+8)) = S^2(-1,\ n/(2n-1),\ (p+1)/2p,\ (p+3)/(2p+8))$. Since $(p+1)/2p + (p+3)/(2p+8) = 1/2 + 1/2p + 1/2 - 1/(2p+8) = 1 + 1/2p - 1/(2p+8) > 1$, $K_{m,p,n}(m_{p,n})$ is an L-space by Lemma 2.3(2).

Finally assume that $m \le -3$. then $K_{m,p,n}(m_{p,n}) = S^2(-n/(mn+1),\ (-p+1)/2p,\ (p-2m-1)/(2p-4m)) = S^2(-1,\ -n/(mn+1),\ (p+1)/2p,\ (p-2m-1)/(2p-4m))$. If $n = 0$, then $K_{m,p,n}(m_{p,n})$ is a lens space, but it is not $S^2 \times S^1$, because $m_{p,n} =$

$m+n(m-p)^2 = m \le -3$. Assume $n \neq 0$. Then by the assumption $p \ge 3, m \le -3$ we have $0 < -n/(mn+1) < 1$, $0 < (p+1)/2p < 1$ and $0 < (p-2m-1)/(2p-4m) = 1/2 - 1/(2p-4m) < 1$. Since $(p+1)/2p + (p-2m-1)/(2p-4m) = 1/2 + 1/2p + 1/2 - 1/(2p-4m) = 1 + 1/2p - 1/(2p-4m) > 1$, Lemma 2.3(2) shows that $K_{m,p,n}(m_{p,n})$ is an L-space.

(2) Since $m \le 0$ and $p \ge 3$, $p \neq 2m \pm 1$, hence $O \cup c_{m,p}$ is a hyperbolic link; see [13, Theorem 6.21]. Then $K_{m,p,n}$ is nontrivial for any $n \neq 0$ [30, 35].

(3) By (1) $K_{m,p,n}$ is an L-space knot. Since $O \cup c_{m,p}$ is a hyperbolic link, the hyperbolicity of $K_{m,p,n}$ for $|n| > 1$ follows from [2, 21, 37]. Thus $K_{m,p,n}$ ($|n| > 1$) is a hyperbolic L-space knot. Let us choose $c_{m,p}$ and then apply n–twist along $c_{m,p}$ to obtain a knot $K_{m,p,n}$. It remains to show that $K_{m,p,n}$ and $K_{m,p,n'}$ are distinct knots. Suppose that $K_{m,p,n}$ and $K_{m,p,n'}$ are isotopic for some integers n and n' with $|n|, |n'| > 1$. Then $(m+n(p-m)^2)$–, and $(m+n'(p-m)^2)$–surgeries on $K_{m,p,n} = K_{m,p,n'}$ produce small Seifert fiber spaces, where $p-m \ge 3$. (Note that since $|n| > 1$, $mn+1$ cannot be zero.) Since $K_{m,p,n}$ is a hyperbolic knot, Lackenby and Meyerhoff [32, Theorem 1.2] prove that the distance $|m+n(p-m)^2 - (m+n'(p-m)^2|$ between above two non-hyperbolic surgeries is at most 8. Hence $|(n-n')(p-m)^2| \le 8$, which implies $n = n'$ because $p-m \ge 3$. □(Proposition 7.1)

Next we investigate link types of $O \cup c_{m,p}$.

Proposition 7.2 *Let $c_{m,p}$ and $c_{m',p'}$ be seiferters for (O,m) and (O,m'), respectively. Suppose that $m, m' \le 0$, $p, p' \ge 3$.*

(1) *If $p-m \neq p'-m'$, then $O \cup c_{m,p}$ and $O \cup c_{m',p'}$ are not isotopic. In particular, if $p \neq p'$, then $O \cup c_{m,p}$ and $O \cup c_{m,p'}$ are not isotopic.*

(2) *If $p-m = p'-m'$, then $O \cup c_{m,p}$ and $O \cup c_{m',p'}$ are not isotopic provided that $|m-m'| > 3$.*

Proof of Proposition 7.2. (1) Note that the linking number between $c_{m,p}$ and O is $p-m$. Hence if $O \cup c_{m,p}$ is isotopic to $O \cup c_{m',p'}$ as ordered links, then we have $p-m = p'-m'$.

(2) Since $p \neq 2m \pm 1$ and $p' \neq 2m' \pm 1$, both $O \cup c_{m,p}$ and $O \cup c_{m',p'}$ are hyperbolic links [13]. Recall that $c_{m,p}$ is a seiferter for (O,m) and $c_{m',p'}$ is a seiferter for (O,m'). Suppose that $O \cup c_{m,p}$ and $O \cup c_{m',p'}$ are isotopic. Then $c_{m,p}$ is a seiferter for (O,m') as well. Let V be the solid torus $S^3 - \mathrm{int}N(c_{m,p})$, which contains O in its interior.

Note that m–surgery of V along O yields a Seifert fiber space over the disk with two exceptional fibers of indices $2p, 2p-4m$, and m'–surgery of V along O yields a Seifert fiber space over the disk with two exceptional fibers of indices $2p', 2p'-4m'$. Since these Seifert fiber spaces contain essential annuli, Gordon and Wu [22, Corollary 1.2] show that $|m-m'| \leq 3$. □(Proposition 7.2)

Theorem 1.10 follows from Propositions 7.1 and 7.2. □(Theorem 1.10)

Remark 7.3 For each seiferter $c_{m,p}$ $(m \leq 0,\ p \geq 3)$, $M_{c_{m,p}}(O,m)$ is an L-space. In fact, $M_{c_{m,p}}(O,m)$, which is the limit of $K_{m,p,n}(m_{p,n})$ when $|n|$ tends to ∞ (Remark 3.2), is $S^2(-1/m, (-p+1)/2p, (p-2m-1)/(2p-4m)) = S^2(-1, -1/m, (p+1)/2p, (p-2m-1)/(2p-4m))$. If $m=-1,0$, then $M_{c_{m,p}}(O,m)$ is an L-space (Claim 3.4). If $m<-1$, since $(p+1)/2p + (p-2m-1)/(2p-4m) = 1 + 1/2p - 1/(2p-4m) > 1$, $M_{c_{m,p}}(O,m)$ is an L-space.

On the other hand, for instance, $M_{c_{3,3}}(O,3)$ is not an L-space. Indeed, $M_{c_{3,3}}(O,3) = S^2(-1/3, -1/3, 2/3) = S^2(-2, 2/3, 2/3, 2/3)$, and taking $k=2$, $a=1$ in Theorem 2.1(3), we have $(1-2/3, 1-2/3, 1-2/3) = (1/3, 1/3, 1/3) < (1/2, 1/2, 1/2)$. Thus $M_{c_{3,3}}(O,3)$ is not an L-space.

8 Hyperbolic, L-space knots with tunnel number greater than one

The purpose in this section is to exhibit infinitely many hyperbolic L-space knots with tunnel number greater than one (Theorem 1.13). In [16] Eudave-Muñoz, Jasso and Miyazaki and the author gave Seifert fibered surgeries which do not arise from primitive/Seifert-fibered construction [10].

Let us take unknotted circles c_a and c_b in $S^3 - T_{3,2}$ as illustrated by Figure 8.1. Then as shown in [16] $\{c_a, c_b\}$ is a *pair of seiferters* for $(T_{3,2}, 7)$, i.e. c_a and c_b become fibers simultaneously in some Seifert fibration of $T_{3,2}(7)$.

Note that the pair $\{c_a,\ c_b\}$ forms the $(4,2)$–torus link in S^3. Hence (-1)–twist along c_a converts $c_a \cup c_b$ into the $(-4,2)$–torus link. Then we can successively apply 1–twist along c_b to obtain $(4,2)$–torus link $c_a \cup c_b$. We denote the images of $c_a,\ c_b$ under twistings along these components by the same symbols $c_a,\ c_b$, respectively.

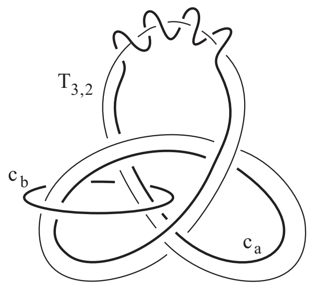

Figure 8.1: $\{c_a, c_b\}$ is a pair of seiferters for $(T_{3,2}, 7)$.

Let $K_{n,0}$ be a knot obtained from $T_{3,2}$ after the sequence of twistings:

$$(c_a, (-1)\text{−twist}) \to (c_b, 1\text{−twist}) \to (c_a, n\text{−twist})$$

Then $K_{n,0} = K(2, -n, 1, 0)$ in [16, Proposition 4.11]. See Figure 8.2.

Similarly, let $K_{0,n}$ be a knot obtained from $T_{3,2}$ after the sequence of twistings:

$$(c_a, (-1)\text{−twist}) \to (c_b, n+1\text{−twist})$$

Then $K_{0,n} = K(2, 0, 1, -n)$ in [16, Proposition 4.11]. See Figure 8.2.

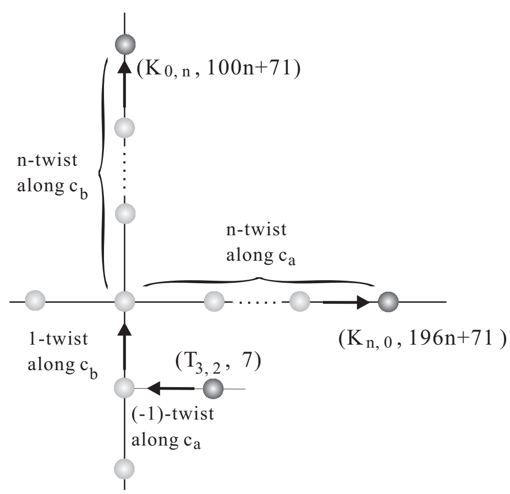

Figure 8.2: Seifert surgeries $(K_{n,0}, 196n+71)$ and $(K_{0,n}, 100n+71)$; each vertex corresponds to a Seifert surgery and each edge corresponds to a single twist along a seiferter.

Theorem 1.13 follows from Theorem 8.1 below.

Theorem 8.1 (1) *$\{K_{n,0}\}_{n\in\mathbb{Z}}$ is a set of mutually distinct hyperbolic L-space knots with tunnel number two.*

(2) *$\{K_{0,n}\}_{n\in\mathbb{Z}\setminus\{-1\}}$ is a set of mutually distinct hyperbolic L-space knots with tunnel number two.*

Proof of Theorem 8.1. We begin by recalling the following result which is a combination of Propositions 3.2, 3.7 and 3.11 in [16].

Lemma 8.2 (1) *$K_{n,0}$ is a hyperbolic knot with tunnel number two, and $K_{n,0}(196n+71)$ is a Seifert fiber space $S^2((11n+4)/(14n+5),\ -2/7,\ 1/2)$.*

(2) *$K_{0,n}$ is a hyperbolic knot with tunnel number two if $n \neq -1$, and $K_{0,n}(100n+71)$ is a Seifert fiber space $S^2(-(3n+2)/(10n+7),\ 4/5,\ 1/2)$.*

Lemma 8.3 (1) *If $K_{n,0}$ and $K_{n',0}$ are isotopic, then $n = n'$.*

(2) *If $K_{0,n}$ and $K_{0,n'}$ are isotopic, then $n = n'$.*

Proof of Lemma 8.3. (1) Suppose that $K_{n,0}$ is isotopic to $K_{n',0}$. Then $K_{n,0}(196n+71)$ and $K_{n,0}(196n'+71)$ are both Seifert fiber spaces. Since $K_{n,0}$ is hyperbolic, Theorem 1.2 in [32] implies that $|196n+71-(196n'+71)| = |196(n-n')| \leq 8$. Hence we have $n = n'$. (2) follows in a similar fashion. □(Lemma 8.3)

Let us prove that $K_{n,0}$ and $K_{0,n}$ are L-space knots for any integer n.

Lemma 8.4 (1) *$K_{n,0}(196n+71)$ is an L-space for any integer n.*

(2) *$K_{0,n}(100n+71)$ is an L-space for any integer n.*

Proof of Lemma 8.4. (1) Note that $K_{n,0}(196n+71) = S^2((11n+4)/(14n+5),\ -2/7,\ 1/2) = S^2(-1, (11n+4)/(14n+5),\ 5/7,\ 1/2)$. Since $0 < (11n+4)/(14n+5) < 1$ for any $n \in \mathbb{Z}$ and $5/7+1/2 \geq 1$, Lemma 2.3(2) shows that $K_{n,0}(196n+71)$ is an L-space for any integer n. This proves (1).

(2) As above first we note that $K_{0,n}(100n+71) = S^2(-(3n+2)/(10n+7),\ 4/5,\ 1/2) = S^2(-1,\ (7n+5)/(10n+7),\ 4/5,\ 1/2)$. Since $0 < (7n+5)/(10n+7) < 1$ for any $n \in \mathbb{Z}$ and $4/5+1/2 \geq 1$, Lemma 2.3(2) shows that $K_{0,n}(100n+71)$ is an L-space for any integer n. □(Lemma 8.4)

Now Theorem 8.1 follows from Lemmas 8.2, 8.3 and 8.4. □(Theorem 8.1)

Question 8.5 Does there exist a hyperbolic L-space knot with tunnel number greater than two? More generally, for a given integer p, does there exist a hyperbolic L-space knot with tunnel number greater than p?

9 Questions

9.1 Characterization of twistings which yield infinitely many L-space knots.

For knots K with Seifert surgery (K, m), Theorems 1.4, 1.5, 1.6 and Corollary 1.7 characterize seiferters which enjoy the desired property in Question 1.1.

The next proposition, which is essentially shown in [25, 26], describes yet another example of twistings which yield infinitely many L-space knots.

Proposition 9.1 (**L-space twisted satellite knots**) *Let k be a nontrivial knot with L-space surgery $(k, 2g-1)$, where g denotes the genus of k, and K a satellite knot of k which lies in $V = N(k)$ with winding number w. Suppose that $V(K; m)$ is a solid torus for some integer $m \geq w^2(2g-1)$. Let c be the boundary of a meridian disk of V, and K_n a knot obtained from K by n–twist along c. Then K_n is an L-space knot for any $n \geq 0$. See Figure 9.1.*

Proof of Proposition 9.1. Recall that $K_n(m+nw^2) = k((m+nw^2)/w^2) = k(m/w^2+n)$ [20]. Since $k(2g-1)$ is an L-space and $m/w^2 \geq 2g-1$, [48, Proposition 9.6] ensures that $k(m/w^2+n)$ is also an L-space if $n \geq 0$. Hence K_n is an L-space knot provided $n \geq 0$. □(Proposition 9.1)

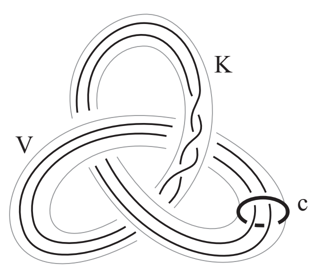

Figure 9.1: K_n is a knot obtained from K by n–twist along c.

Remark 9.2 (1) In Proposition 9.1, the knot K in the solid torus V is required to have a cosmetic surgery: $V(K; m) \cong S^1 \times D^2$. The cosmetic surgery of the solid torus is well-understood by [18, 5].

(2) Twisting operation described in Proposition 9.1 can be applied only for satellite knots and resulting knots after the twistings are also satellite knots.

(3) In Proposition 9.1, the knot k is assumed to be nontrivial. If k is a trivial knot in S^3, then $K(m) = (S^3 - \mathrm{int}V) \cup V(K;m)$ is a lens space, hence (K,m) is an L-space surgery. It is easy to see that c is a seiferter for (K,m).

For further possibility, weaken a condition of seiferter to obtain a notion of "pseudo-seiferter" as follows.

Definition 9.3 (**pseudo-seiferter**) Let (K,m) be a Seifert surgery. A knot c in $S^3 - N(K)$ is called a *pseudo-seiferter* for (K,m) if c satisfies (1) and (2) below.

(1) c is a trivial knot in S^3.

(2) c becomes a "cable" of a fiber in a Seifert fibration of $K(m)$ and the preferred longitude λ of c in S^3 becomes the cabling slope of c in $K(m)$.

We do not know if a pseudo-seiferter exists, but if (K,m) admits a pseudo-seiferter, it behaves like a seiferter in the following sense. Let V be a fibered tubular neighborhood of a fiber t and c is a cable in V. Then the result of a surgery (corresponding to n–twist) on c of V is again a solid torus, and this surgery is reduced to a surgery on the fiber t which is a core of V. Hence $K_n(m_n)$ is a (possibly degenerate) Seifert fiber space. This suggests that a pseudo-seiferter is also a candidate for an unknotted circle described in Question 1.1.

We would like to ask the following question for non-satellite knots.

Question 9.4 Let K be a non-satellite knot and K_n a knot obtained from K by n–twist along an unknotted circle c in $S^3 - K$. Suppose that the twist family $\{K_n\}$ contains infinitely many L-space knots.

(1) Does K admit a Seifert surgery (K,m) for which c is a seiferter?

(2) Does K admit a Seifert surgery (K,m) for which c is a seiferter or a pseudo-seiferter?

9.2 L-space knots and strong invertibility.

A knot is said to be *strongly invertible* if there exists an orientation preserving involution of S^3 which fixes the knot setwise and reverses orientation. Known L-space knots are strongly invertible, so it is natural to ask:

Problem 9.5 (Watson) Are L-space knots strongly invertible?

In [13] an “asymmetric seiferter" defined below is essentially used to find Seifert fibered sugary on knots with no symmetry.

Definition 9.6 (**asymmetric seiferter**) A seiferter c for a Seifert surgery (K, m) is said to be *symmetric* if we have an orientation preserving diffeomorphism $f : S^3 \to S^3$ of finite order with $f(K) = K, f(c) = c$; otherwise, c is called an *asymmetric seiferter*.

Combining [13, Theorem 7.3] and Theorem 1.4, we obtain:

Proposition 9.7 *Let (K, m) be a Seifert fibered surgery on a non-satellite knot with an asymmetric seiferter c which becomes an exceptional fiber. Suppose that $M_c(K, m)$ is an L-space. Then there is a constant N such that K_n, a knot obtained from K by n–twist along c, is a hyperbolic L-space knot with no symmetry for any $n \leq N$ or $n \geq N$.*

If c is a seiferter for $(T_{p,q}, pq)$ which becomes a degenerate fiber in $T_{p,q}(pq)$, then c is a meridian of $T_{p,q}$ or $T_{p,q} \cup c$ is a hyperbolic link in S^3; see [13, Theorem 3.19(3)]. Hence the argument in the proof of Theorem 7.3 in [13] and Theorem 1.6(2) enable us to show:

Proposition 9.8 *If c is an asymmetric seiferter for $(T_{p,q}, pq)$ which becomes a degenerate fiber in $T_{p,q}(pq)$, then there is a constant N such that K_n is a hyperbolic L-space knot with no symmetry for any $n \leq N$ or $n \geq N$.*

For the asymmetric seiferter $c = c_1'$ for $(K, m) = (P(-3, 3, 5), 1)$ given in [13, Lemma 7.5], $M_c(K, m)$ is not an L-space and c does not satisfy the hypothesis of Proposition 9.7.

Question 9.9 Does there exist an asymmetric seiferter described in Propositions 9.7 and 9.8?

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