L-space surgery and twisting operation

KIMIHIKO MOTEKI

A knot in the 3–sphere is called an L-space knot if it admits a nontrivial Dehn surgery yielding an L-space, i.e. a rational homology 3–sphere with the smallest possible Heegaard Floer homology. Given a knot $K$, take an unknotted circle $c$ and twist $K$ $n$ times along $c$ to obtain a twist family $\{K_n\}$. We give a sufficient condition for $\{K_n\}$ to contain infinitely many L-space knots. As an application we show that for each torus knot and each hyperbolic Berge knot $K$, we can take $c$ so that the twist family $\{K_n\}$ contains infinitely many hyperbolic L-space knots. We also demonstrate that there is a twist family of hyperbolic L-space knots each member of which has tunnel number greater than one.

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1 Introduction

Heegaard Floer theory (with $\mathbb{Z}/2\mathbb{Z}$ coefficients) associates a group $\widehat{HF}(M, t)$ to a closed, orientable spin$^c$ 3–manifold $(M, t)$. The direct sum of $\widehat{HF}(M, t)$ for all spin$^c$ structures is denoted by $\widehat{HF}(M)$. A rational homology 3–sphere $M$ is called an L-space if $\widehat{HF}(M, t)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ for all spin$^c$ structure $t \in \text{Spin}^c(M)$. Equivalently, the dimension of $\dim_{\mathbb{Z}/2\mathbb{Z}} \widehat{HF}(M)$ is equal to the order of $|H_1(M; \mathbb{Z})|$. A knot $K$ in the 3–sphere $S^3$ is called an L-space knot if the result $K(r)$ of $r$–surgery on $K$ is an L-space for some non-zero integer $r$, and the pair $(K, r)$ is called an L-space surgery. The class of L-spaces includes lens spaces (except $S^2 \times S^1$), and more generally, 3–manifolds with elliptic geometry [47, Proposition 2.3]. Since the trivial knot, nontrivial torus knots and Berge knots [6] admit nontrivial surgeries yielding lens spaces, these are fundamental examples of L-space knots. For the mirror image $K^*$ of $K$, $K^*(-r)$ is orientation reversingly homeomorphic to $K(r)$. So if $K(r)$ is an L-space, then $K^*(-r)$ is also an L-space [47, p.1288]. Hence if $K$ is an L-space knot, then so is $K^*$.

Let $K$ be a nontrivial L-space knot with a positive L-space surgery, then Ozsváth and Szabó [48, Proposition 9.6] ([25, Lemma 2.13]) prove that $r$–surgery on $K$ results in an L-space if and only if $r \geq 2g(K) - 1$, where $g(K)$ denotes the genus of $K$. This result, together with Thurston’s hyperbolic Dehn surgery theorem [51, 52, 4, 49, 7], shows
that each hyperbolic L-space knot, say a hyperbolic Berge knot, produces infinitely many hyperbolic L-spaces by Dehn surgery.

On the other hand, there are some strong constraints for L-space knots:

- The non-zero coefficients of the Alexander polynomial of an L-space knot are ±1 and alternate in sign [47, Corollary 1.3].
- An L-space knot is fibered [43, Corollary 1.2][[44]]; see also [19, 29].
- An L-space knot is prime [31, Theorem 1.2].

Note that these conditions are not sufficient. For instance, 10_{132} satisfies the above conditions, but it is not an L-space knot; see [47].

As shown in [25, 26], some satellite operations keep the property of being L-space knots. In the present article, we consider if some suitably chosen twistings also keep the property of being L-space knots. Given a knot $K$, take an unknotted circle $c$ which bounds a disk intersecting $K$ at least twice. Then performing $n$–twist, i.e. $(-1/n)$–surgery along $c$, we obtain another knot $K_n$. Then our question is formulated as:

**Question 1.1** Which knots $K$ admit an unknotted circle $c$ such that $n$–twist along $c$ converts $K$ into an L-space knot $K_n$ for infinitely many integers $n$? Furthermore, if $K$ has such a circle $c$, which circles enjoy the desired property?

**Example 1.2** Let $K$ be a pretzel knot $P(-2, 3, 7)$ and take an unknotted circle $c$ as in Figure 1.1. Then following Ozsváth and Szabó [47] $K_n$ is an L-space knot if $n \geq -3$ and thus the twist family $\{K_n\}$ contains infinity many L-space knots. Note that this family, together with a twist family $\{T_{2n+1,2}\}$, comprise all Montesinos L-space knots; see [33] and [3].

In this example, it turns out that $c$ becomes a Seifert fiber in the lens space $K(19)$ (cf. Example 4.3). We employed such a circle for relating Seifert fibered surgeries in [13]. A pair $(K, m)$ of a knot $K$ in $S^3$ and an integer $m$ is a Seifert surgery if $K(m)$ has a Seifert fibration; we allow the fibration to be degenerate, i.e. it contains an exceptional fiber of index 0 as a degenerate fiber. See [13, 2.1] for details. The definition below enables us to say that $c$ is a seiferter for the Seifert (lens space) surgery $(K, 19)$.

**Definition 1.3** (seiferter [13]) Let $(K, m)$ be a Seifert surgery. A knot $c$ in $S^3 - N(K)$ is called a seiferter for $(K, m)$ if $c$ satisfies the following:
Figure 1.1: A knot $K_n$ obtained by $n$–twist along $c$ is an L-space knot if $n \geq -3$.

- $c$ is a trivial knot in $S^3$.
- $c$ becomes a fiber in a Seifert fibration of $K(m)$.

As remarked in [13, Convention 2.15], if $c$ bounds a disk in $S^3 - K$, then we do not regard $c$ as a seiferter. Thus for any seiferter $c$ for $(K, m)$, $S^3 - \text{int} N(K \cup c)$ is irreducible.

Let $(K, m)$ be a Seifert surgery with a seiferter $c$. There are two cases according as $c$ becomes a fiber in a non-degenerate Seifert fibration of $K(m)$ or $c$ becomes a fiber in a degenerate Seifert fibration of $K(m)$. In the former case, for homological reasons, the base surface is the 2–sphere $S^2$ or the projective plane $\mathbb{R}P^2$. Suppose that $c$ is a fiber in a non-degenerate Seifert fibration of $K(m)$ over the 2–sphere $S^2$. Then in the following we assume that $K(m)$ contains at most three exceptional fibers and if there are three exceptional fibers, then $c$ is an exceptional fiber. We call such a seiferter a seiferter for a small Seifert fibered surgery $(K, m)$. To be precise, the images of $K$ and $m$ after $n$–twist along $c$ should be denoted by $K_{c,n}$ and $m_{c,n}$, but for simplicity, we abbreviate them to $K_n$ and $m_n$ respectively as long as there is no confusion.

**Theorem 1.4** Let $c$ be a seiferter for a small Seifert fibered surgery $(K, m)$. Then $(K_n, m_n)$ is an L-space surgery for an infinite interval of integers $n$ if and only if the result of $(m, 0)$–surgery on $K \cup c$ is an L-space.

In remaining cases, it turns out that every seiferter enjoys the desired property in Question 1.1.

**Theorem 1.5** Let $c$ be a seiferter for $(K, m)$ which become a fiber in a Seifert fibration of $K(m)$ over $\mathbb{R}P^2$. Then $(K_n, m_n)$ is an L-space surgery for all but at most one integer $n_0$ with $(K_{n_0}, m_{n_0}) = (O, 0)$. Hence $K_n$ is an L-space knot for all integers $n$. 
Let us turn to the case where \( c \) is a (degenerate or non-degenerate) fiber in a degenerate Seifert fibration of \( K(m) \). Recall from [13, Proposition 2.8] that if \( K(m) \) has a degenerate Seifert fibration, then it is a lens space or a connected sum of two lens spaces such that each summand is neither \( S^3 \) nor \( S^2 \times S^1 \). The latter 3–manifold will be simply referred to as a connected sum of two lens spaces, which is an L-space [50, 8.1(5)] ([45]).

**Theorem 1.6** Let \( c \) be a seiferter for \( (K, m) \) which becomes a (degenerate or non-degenerate) fiber in a degenerate Seifert fibration of \( K(m) \).

1. If \( K(m) \) is a lens space, then \( (K_n, m_n) \) is an L-space surgery, hence \( K_n \) is an L-space knot, for all but at most one integer \( n \).
2. If \( K(m) \) is a connected sum of two lens spaces, then \( (K_n, m_n) \) is an L-space surgery, hence \( K_n \) is an L-space knot, for any integer \( n \geq -1 \) or \( n \leq 1 \).

Following Greene [23, Theorem 1.5], if \( K(m) \) is a connected sum of two lens spaces, then \( K \) is a torus knot \( T_{p,q} \) or a cable of a torus knot \( C_{p,q}(T_{r,s}) \), where \( p = qrs \pm 1 \). We may assume \( p, q \geq 2 \) by taking the mirror image if necessary. The next theorem is a refinement of Theorem 1.6(2).

**Theorem 1.7** Let \( c \) be a seiferter for \( (K, m) = (T_{p,q}, pq) \) or \( (C_{p,q}(T_{r,s}), pq) \) \( (p = qrs \pm 1) \). We assume \( p, q \geq 2 \). Then a knot \( K_n \) obtained from \( K \) by \( n \)-twist along \( c \) is an L-space knot for any \( n \geq -1 \). Furthermore, if the linking number \( l \) between \( c \) and \( K \) satisfies \( l^2 \geq 2pq \), then \( K_n \) is an L-space knot for all integers \( n \).

In the above theorem, even when \( l^2 < 2pq \), \( K_n \) \( (n < -1) \) may be an L-space knot; see [41].

In Sections 5, 6 and 7 we will exploit seiferter technology developed in [13, 11, 12] to give a partial answer to Question 1.1. Even though Theorem 1.7 treats a special kind of Seifert surgeries, it offers many applications. In particular, Theorem 1.7 enables us to give new families of L-space twisted torus knots. See Section 5 for the definition of twisted torus knots \( K(p, q; r, n) \) introduced by Dean [10].

**Theorem 1.8** (L-space twisted torus knots) (1) The following twisted torus knots are L-space knots for all integers \( n \).

- \( K(p, q; p + q, n) \) with \( p, q \geq 2 \)
- \( K(3p + 1, 2p + 1; 4p + 1, n) \) with \( p > 0 \)
- \( K(3p + 2, 2p + 1; 4p + 3, n) \) with \( p > 0 \)
The following twisted torus knots are L-space knots for any $n \geq -1$.

- $K(p, q; p - q, n)$ with $p, q \geq 2$
- $K(2p + 3, 2p + 1; 2p + 2, n)$ with $p > 0$

Theorem 1.8 has the following corollary, which asserts that every nontrivial torus knot admits twistings desired in Question 1.1.

**Corollary 1.9** For any nontrivial torus knot $T_{p,q}$, we can take an unknotted circle $c$ so that $n$–twist along $c$ converts $T_{p,q}$ into an L-space knot $K_n$ for all integers $n$. Furthermore, $\{K_n\}_{|n|\geq 3}$ is a set of mutually distinct hyperbolic L-space knots.

For the simplest L-space knot, i.e. the trivial knot $O$, we can strengthen Corollary 1.9 as follows.

**Theorem 1.10** (L-space twisted unknots) For the trivial knot $O$, we can take infinitely many unknotted circles $c$ so that $n$–twist along $c$ changes $O$ into a nontrivial L-space knot $K_{c,n}$ for any non-zero integer $n$. Furthermore, $\{K_{c,n}\}_{|n|\geq 1}$ is a set of mutually distinct hyperbolic L-space knots.

Using a relationship between Berge’s lens space surgeries and surgeries yielding a connected sum of two lens spaces, we can prove:

**Theorem 1.11** (L-space twisted Berge knots) For any hyperbolic Berge knot $K$, there is an unknotted circle $c$ such that $n$–twist along $c$ converts $K$ into a hyperbolic L-space knot $K_{c,n}$ for infinitely many integers $n$.

In Section 8 we consider the tunnel number of L-space knots. Recall that the tunnel number of a knot $K$ in $S^3$ is the minimum number of mutually disjoint, embedded arcs connecting $K$ such that the exterior of the resulting 1–complex is a handlebody. Hedden’s cabling construction [25], together with [40], enables us to obtain an L-space knot with tunnel number greater than 1. Actually Baker and Moore [3] have shown that for any integer $N$, there is an L-space knot with tunnel number greater than $N$. However, L-space knots with tunnel number greater than one constructed above are all satellite (non-hyperbolic) knots and they ask:

**Question 1.12** ([3]) Is there a non-satellite, L-space knot with tunnel number greater than one?
Examining knots with Seifert surgeries which do not arise from primitive/Seifert-fibred construction given by [16], we prove the following which answers the question in the positive.

**Theorem 1.13** There exist infinitely many hyperbolic L-space knots with tunnel number greater than one.

Each knot in the theorem is obtained from a trefoil knot $T_{3, 2}$ by alternate twisting along two seiferters for the lens space surgery $(T_{3, 2}, 7)$.

In Section 9 we will discuss further questions on relationships between L-space knots and twisting operation.

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## 2 Seifert fibered L-spaces

Let $M$ be a rational homology 3–sphere which is a Seifert fiber space. For homological reasons, the base surface of $M$ is either $S^2$ or $\mathbb{R}P^2$. In the latter case, Boyer, Gordon and Watson [8, Proposition 5] prove that $M$ is an L-space. Now assume that the base surface of $M$ is $S^2$. Following Ozsváth and Szabó [46, Theorem 1.4] if $M$ is an L-space, then it carries no taut foliation, in particular, it carries no horizontal (i.e. transverse) foliation. Furthermore, Lisca and Stipsicz [34, Theorem 1.1] prove that the converse also does hold. Therefore a Seifert fibered rational homology 3–sphere $M$ over $S^2$ is an L-space if and only if it does not admit a horizontal foliation. Note that if $M$ does not carry a horizontal foliation, then it is necessarily a rational homology 3–sphere. In fact, if $|H_1(M; \mathbb{Z})| = \infty$, then $M$ is a surface bundle over the circle [27, VI.34], [24], and hence it has a horizontal foliation. On the other hand, Eisenbud-Hirsh-Neumann
[14], Jankins-Neumann [28] and Naimi [42] gave a necessary and sufficient conditions for a Seifert fibered 3-manifold to carry a horizontal foliation. Combining them we have Theorem 2.1 below. See also [9, Theorem 5.4]; we follow the convention of Seifert invariants in [9, Section 4].

For ordered triples \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\), we write \((a_1, a_2, a_3) < (b_1, b_2, b_3)\) (resp. \((a_1, a_2, a_3) \leq (b_1, b_2, b_3)\)) if \(a_i < b_i\) (resp. \(a_i \leq b_i\)) for \(1 \leq i \leq 3\), and denote by \((a_1, a_2, a_3)^*\) the ordered triple \((\sigma(a_1), \sigma(a_2), \sigma(a_3))\), where \(\sigma\) is a permutation such that \(\sigma(a_1) \leq \sigma(a_2) \leq \sigma(a_3)\).

**Theorem 2.1** ([46, 34, 14, 28, 42]) A Seifert fiber space \(S^2(b, r_1, r_2, r_3)\) \((b \in \mathbb{Z}, 0 < r_1 < 1)\) is an L-space if and only if one of the following holds.

1. \(b \geq 0\) or \(b \leq -3\).
2. \(b = -1\) and there are no relatively prime integers \(a, k\) such that \(0 < a \leq k/2\) and \((r_1, r_2, r_3)^* < (1/k, a/k, (k-a)/k)\).
3. \(b = -2\) and there are no relatively prime integers \(0 < a \leq k/2\) such that \((1-r_1, 1-r_2, 1-r_3)^* < (1/k, a/k, (k-a)/k)\).

For our purpose, we consider the following problem:

**Problem 2.2** Given an integer \(b\) and rational numbers \(0 < r_1 \leq r_2 < 1\), describe rational numbers \(-1 \leq r \leq 1\) for which \(S^2(b, r_1, r_2, r)\) is an L-space.

We begin by observing:

**Lemma 2.3** Assume that \(0 < r_1 \leq r_2 < 1\).

1. If \(b \geq 0\) or \(b \leq -3\), then \(S^2(b, r_1, r_2, r)\) is an L-space for any \(0 < r < 1\).
2. If \(r_1 + r_2 \geq 1\), then \(S^2(-1, r_1, r_2, r)\) is an L-space for any \(0 < r < 1\).
3. If \(r_1 + r_2 \leq 1\), then \(S^2(-2, r_1, r_2, r)\) is an L-space for any \(0 < r < 1\).

**Proof of Lemma 2.3.** The first assertion is nothing but Theorem 2.1(1). Suppose for a contradiction that \(S^2(-1, r_1, r_2, r)\) is not an L-space for some \(0 < r < 1\). Then, by Theorem 2.1(2) we can take relatively prime integers \(a, k\) \((0 < a \leq k/2)\) so that \((r_1, r_2, r)^* < (1/k, a/k, (k-a)/k)\). This then implies that \(r_1 < a/k\) and \(r_2 < (k-a)/k\). Hence \(r_1 + r_2 < a/k + (k-a)/k = 1\), a contradiction. This proves (2).
To prove (3), assume for a contradiction that $S^2(-2, r_1, r_2, r)$ is not an L-space for some $0 < r < 1$. Then, by Theorem 2.1(3) we have relatively prime integers $a, k$ ($0 < a \leq k/2$) such that $(1 - r_1, 1 - r_2, 1 - r) < (1/k, a/k, (k-a)/k)$. Thus we have $(1 - r_2) < a/k$ and $(1 - r_1) < (k-a)/k$. Thus $(1 - r_1) + (1 - r_2) < 1$, which implies $r_1 + r_2 > 1$, contradicting the assumption. \hfill \square (Lemma 2.3)

Now let us prove the following, which gives an answer to Problem 2.2.

**Proposition 2.4** Assume that $0 < r_1 \leq r_2 < 1$.

1. If $b \leq -3$ or $b \geq 1$, then $S^2(b, r_1, r_2, r)$ is an L-space for any $-1 \leq r \leq 1$.
2. If $b = -2$, then there exists $\varepsilon > 0$ such that $S^2(-2, r_1, r_2, r)$ is an L-space for any $-1 \leq r \leq \varepsilon$. Furthermore, if $r_1 + r_2 \leq 1$, then $S^2(-2, r_1, r_2, r)$ is an L-space if $-1 < r < 1$.
3. Suppose that $b = -1$.
   (i) If $r_1 + r_2 \geq 1$, then $S^2(-1, r_1, r_2, r)$ is an L-space for any $0 < r \leq 1$.
   (ii) If $r_1 + r_2 \leq 1$, then $S^2(-1, r_1, r_2, r)$ is an L-space for any $-1 < r < 0$. 
4. If $b = 0$, then there exists $\varepsilon > 0$ such that $S^2(r_1, r_2, r)$ is an L-space for any $-\varepsilon \leq r \leq 1$. Furthermore, if $r_1 + r_2 \geq 1$, then $S^2(r_1, r_2, r)$ is an L-space if $-1 < r \leq 1$.

**Proof of Proposition 2.4.** If $r = 0, \pm 1$, then $S^2(b, r_1, r_2, r)$ is a lens space.

**Claim 2.5** Suppose that $r$ is an integer. Then the lens space $S^2(b, r_1, r_2, r)$ is $S^2 \times S^1$ if and only if $b + r = -1$ and $r_1 + r_2 = 1$. In particular, if $b + r \neq -1$, then $S^2(b, r_1, r_2, r)$ is an L-space.

**Proof of Claim 2.5.** Recall that $H_1(S^2(a/b, c/d)) \cong \mathbb{Z}$ $(b, d \geq 1)$ if and only if $ad + bc = 0$, i.e. $a/b + c/d = 0$. Thus $S^2(b, r_1, r_2, r)$ is $S^2 \times S^1$ if and only if $b + r_1 + r_2 + r = 0$, i.e. $r_1 + r_2 = -b - r \in \mathbb{Z}$. Since $0 < r_i < 1$, we have $r_1 + r_2 = 1$ and $b + r = -1$. \hfill \square (Claim 2.5)

We divide into two cases according as $0 \leq r \leq 1$ or $-1 \leq r \leq 0$.

**Case I.** $0 \leq r \leq 1$.

(i) If $b \geq 0$ or $b \leq -3$, then $S^2(b, r_1, r_2, r)$ is an L-space for any $0 < r < 1$ by Lemma 2.3(1). Since $b + r \neq -1$ for $r = 0, 1$, by Claim 2.5 $S^2(b, r_1, r_2, r)$ is an L-space for $r = 0, 1$. Hence $S^2(b, r_1, r_2, r)$ is an L-space for any $0 \leq r \leq 1$. 


(ii) Suppose that $b = -1$. By Lemma 2.3(2), if $r_1 + r_2 \geq 1$, then $S^2(-1, r_1, r_2, r)$ is an L-space for any $0 < r < 1$. Since $S^2(-1, r_1, r_2, 1)$ is an L-space (Claim 2.5), $S^2(-1, r_1, r_2, r)$ is an L-space for any $0 < r \leq 1$.

(iii) Assume $b = -2$. Let us assume $0 < r \leq r_1$ so that $0 < 1 - r_2 \leq 1 - r_1 \leq 1 - r < 1$. Set $A = \{(k - a)/k \mid 1 - r_2 < 1/k, 1 - r_1 < a/k, 0 < a \leq k/2, a$ and $k$ are relatively prime integers}. If $A = \emptyset$, i.e. there are no relatively prime integers $a, k (0 < a \leq k/2)$ such that $1 - r_2 < 1/k, 1 - r_1 < a/k$, then $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 < r \leq r_1$ by Theorem 2.1. Suppose that $A \neq \emptyset$. Since there are only finitely many integers $k$ satisfying $1 - r_2 < 1/k$, $A$ consists of only finitely many elements. Let $r_0$ be the maximal element in $A$. If $0 < r \leq 1 - r_0$, then $r_0 \leq 1 - r < 1$, and hence there are no relatively prime integers $a, k (0 < a \leq k/2)$ satisfying $(1 - r_2, 1 - r_1, 1 - r) < (1/k, a/k, (k - a)/k)$. Put $\varepsilon = \min\{r_1, 1 - r_0\}$. Then $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 < r \leq \varepsilon$ by Theorem 2.1. Since $S^2(-2, r_1, r_2, 0)$ is an L-space (Claim 2.5), $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 \leq r \leq \varepsilon$. Furthermore, if we have the additional condition $r_1 + r_2 \leq 1$, then Lemma 2.3(3) improves the result so that $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 \leq r < 1$.

**Case II.** $-1 \leq r \leq 0$.

Note that $S^2(b, r_1, r_2, r) = S^2(b - 1, r_1, r_2, r + 1)$.

(i) If $b \geq 1$ or $b \leq -2$ (i.e. $b - 1 \geq 0$ or $b - 1 \leq -3$), then $S^2(b, r_1, r_2, r) = S^2(b - 1, r_1, r_2, r + 1)$ is an L-space for any $0 < r + 1 < 1$, i.e. $-1 < r < 0$ by Lemma 2.3(1). Since $b + r \neq -1$ for $r = -1, 0$, $S^2(b, r_1, r_2, r)$ is an L-space for $r = -1, 0$ (Claim 2.5). Thus $S^2(b, r_1, r_2, r)$ is an L-space for any $-1 \leq r \leq 0$.

(ii) If $b = 0$ (i.e. $b - 1 = -1$), then $S^2(0, r_1, r_2, r) = S^2(-1, r_1, r_2, r + 1)$. Let us assume $r_2 - 1 \leq r < 0$ so that $0 < r_1 \leq r + 1 < 1$. Set $A = \{(k - a)/k \mid r_1 < 1/k, r_2 < a/k, 0 < a \leq k/2, a$ and $k$ are relatively prime integers}. If $A = \emptyset$, then we can easily observe that for any $r$ with $r_2 \leq r + 1 < 1$, $S^2(-1, r_1, r_2, r + 1)$ is an L-space (Theorem 2.1). Hence for any $r_2 - 1 \leq r < 0$, $S^2(0, r_1, r_2, r)$ is an L-space. Suppose that $A \neq \emptyset$. Since $A$ is a finite set, we take the maximal element $r_0$ in $A$. If $r_0 \leq r + 1 < 1$ (i.e. $r_0 - 1 \leq r < 0$), then there are no relatively prime integers $a, k (0 < a \leq k/2)$ satisfying $(r_1, r_2, r + 1) < (1/k, a/k, (k - a)/k)$. Put $\varepsilon = \min\{1 - r_2, 1 - r_0\}$. Then $S^2(0, r_1, r_2, r) = S^2(-1, r_1, r_2, r + 1)$ is an L-space for any $-\varepsilon \leq r < 0$ (Theorem 2.1). Since $S^2(0, r_1, r_2, 0) = S^2(r_1, r_2)$ is an L-space (Claim 2.5), $S^2(0, r_1, r_2, r)$ is an L-space for any $-\varepsilon \leq r \leq 0$. Furthermore, if we
have the additional condition $r_1 + r_2 \geq 1$, then Lemma 2.3(2) improves the result so that $S^2(r_1, r_2, r) = S^2(-1, r_1, r_2, r + 1)$ is an L-space for any $-1 < r \leq 0$.

(iii) If $b = -1$ (i.e. $b - 1 = -2$), then $S^2(-1, r_1, r_2, r) = S^2(-2, r_1, r_2, r + 1)$. Assume that $r_1 + r_2 \leq 1$. Then Proposition 2.3(3), $S^2(-1, r_1, r_2, r) = S^2(-2, r_1, r_2, r + 1)$ is an L-space for any $0 < r + 1 < 1$, i.e. $-1 < r < 0$. Since Claim 2.5 shows that $S^2(-1, r_1, r_2, -1)$ is an L-space, $S^2(-1, r_1, r_2, r)$ is an L-space for any $-1 \leq r < 0$.

Combining Cases I and II, we obtain the result described in the proposition.

□ (Proposition 2.4)

The next proposition shows that if $S^2(b, r_1, r_2, r_\infty)$ is an L-space for some rational number $0 < r_\infty < 1$, then we can find $r$ near $r_\infty$ so that $S^2(b, r_1, r_2, r)$ is an L-space.

**Proposition 2.6**  Suppose that $0 < r_1 \leq r_2 < 1$ and $S^2(b, r_1, r_2, r_\infty)$ is an L-space for some rational number $0 < r_\infty < 1$.

(1) If $b = -1$, then $S^2(-1, r_1, r_2, r)$ is an L-space for any $r_\infty \leq r \leq 1$.

(2) If $b = -2$, then $S^2(-2, r_1, r_2, r)$ is an L-space for any $-1 \leq r \leq r_\infty$.

**Proof of Proposition 2.6.** (1) Assume for a contradiction that $S^2(-1, r_1, r_2, r)$ is not an L-space for some $r$ satisfying $r_\infty \leq r < 1$. By Theorem 2.1 we have relatively prime integers $a, k$ $(0 < a \leq k/2)$ such that $(r_1, r_2, a) < (1/k, a/k, (k - a)/k)$. Since $r_\infty \leq r < 1$, $(r_1, r_2, r_\infty)^* \leq (r_1, r_2, r)^* < (1/k, a/k, (k - a)/k)$. Hence Theorem 2.1 shows that $S^2(-1, r_1, r_2, r_\infty)$ is not an L-space, a contradiction. Since $S^2(-1, r_1, r_2, 1)$ is an L-space (Claim 2.5), $S^2(-1, r_1, r_2, r)$ is an L-space for any $r_\infty \leq r \leq 1$.

(2) Next assume for a contradiction that $S^2(-2, r_1, r_2, r)$ is not an L-space for some $r$ satisfying $0 < r \leq r_\infty$. Then following Theorem 2.1 we have $(1 - r_1, 1 - r_2, 1 - r)^* < (1/k, a/k, (k - a)/k)$ for some relatively prime integers $a, k$ $(0 < a \leq k/2)$. Since $r \leq r_\infty$, we have $1 - r_\infty \leq 1 - r$, and hence $(1 - r_1, 1 - r_2, 1 - r_\infty)^* \leq (1 - r_1, 1 - r_2, 1 - r)^* < (1/k, a/k, (k - a)/k)$. This means $S^2(-2, r_1, r_2, r_\infty)$ is not an L-space, contradicting the assumption. Thus $S^2(-2, r_1, r_2, r)$ is an L-space for any $0 < r \leq r_\infty$. Furthermore, as shown in Proposition 2.4(2), $S^2(-2, r_1, r_2, r)$ is an L-space if $-1 \leq r \leq \varepsilon$ for some $\varepsilon > 0$, so $S^2(-2, r_1, r_2, r)$ is an L-space for any $-1 \leq r \leq r_\infty$. □ (Proposition 2.6)

We close this section with the following result which is the complement of Proposition 2.6.
Suppose that shows that there are relatively prime integers $1.4 \leq r_2 < 1$. If $2.7$ and $1.4 > 2.1$, we have relatively prime $2.1 < r_2 < 1$.

### Proof of Proposition 2.7

1. Since $S^2(-1, r_2, r_\infty)$ is not an L-space, Theorem 2.1 shows that there are relatively prime integers $a, k (0 < a \leq k/2)$ such that $(r_1, r_2, r_\infty)^a < (1/k, a/k, (k-a)/k)$. Then clearly there exists $\varepsilon > 0$ such that for any $0 < r < r_\infty + \varepsilon$, we have $(r_1, r_2, r)^a < (1/k, a/k, (k-a)/k)$. Thus by Theorem 2.1 again $S^2(-1, r_1, r_2, r)$ is not an L-space for any $0 < r < r_\infty + \varepsilon$.

2. Since $S^2(-2, r_1, r_2, r_\infty)$ is not an L-space, by Theorem 2.1 we have relatively prime integers $a, k (0 < a \leq k/2)$ such that $(1-r_1, 1-r_2, 1-r_\infty)^a < (1/k, a/k, (k-a)/k)$. Hence there exists $\varepsilon > 0$ such that if $0 < 1-r < 1-r_\infty + \varepsilon$, i.e. $r_\infty - \varepsilon < r < 1$, then $(1-r_1, 1-r_2, 1-r)^a < (1/k, a/k, (k-a)/k)$. Following Theorem 2.1 $S^2(-2, r_1, r_2, r)$ is not an L-space for any $r_\infty - \varepsilon < r < 1$. □ (Proposition 2.7)

### 3 L-space surgeries and twisting along seiferters I – non-degenerate case

The goal in this section is to prove Theorems 1.4 and 1.5.

Let $c$ be a seifier for a small Seifert fibered surgery $(K, m)$. The 3-manifold obtained by $(m, 0)$–surgery on $K \cup c$ is denoted by $M_c(K, m)$.

#### Proof of Theorem 1.4

First we prove the “if” part of Theorem 1.4. If $K(m)$ is a lens space and $c$ is a core of the genus one Heegaard splitting, then $K(m)$ is a lens space for any integer $n$. Thus $(K_n, m_n)$ is an L-space surgery for all $n \in \mathbb{Z}$ except when $K_n(m_n) \cong S^2 \times S^1$, i.e. $K_n$ is the trivial knot and $m_n = 0$ [17, Theorem 8.1]. Since $(K_n, m_n) = (K_{n'}, m_{n'})$ if and only if $n = n'$ [13, Theorem 5.1], there is at most one integer $n$ such that $(K_n, m_n) = (O, 0)$. Henceforth, in the case where $K(m)$ is a lens space, we assume that $K(m)$ has a Seifert fibration over $S^2$ with two exceptional fibers $t_1, t_2$, and $c$ becomes a regular fiber in this Seifert fibration.
Let $E$ be $K(m) - \text{int}N(c)$ with a fibered tubular neighborhood of the union of two exceptional fibers $t_1, t_2$ and one regular fiber $t_0$ removed. Then $E$ is a product circle bundle over the four times punctured sphere. Take a cross section of $E$ such that $K(m)$ is expressed as $S^2(b, r_1, r_2, r_3)$, where the Seifert invariant of $t_0$ is $b \in \mathbb{Z}$, that of $t_i$ is $0 < r_i < 1$ ($i = 1, 2$), and that of $c$ is $0 \leq r_3 < 1$. Without loss of generality, we may assume $r_1 \leq r_2$. Let $s$ be the boundary curve on $\partial N(c)$ of the cross section so that $[s] \cdot [t] = 1$ for a regular fiber $t \subset \partial N(c)$. Let $(\mu, \lambda)$ be a preferred meridian-longitude pair of $c \subset S^3$. Then $[\mu] = \alpha_3[s] + \beta_3[t] \in H_1(\partial N(c))$ and $[\lambda] = -\alpha[s] - \beta[t] \in H_1(\partial N(c))$ for some integers $\alpha_3, \beta_3, \alpha$ and $\beta$ which satisfy $\alpha_3 > 0$ and $\alpha \beta_3 - \beta \alpha_3 = 1$, where $r_3 = \beta_3/\alpha_3$. Now let us write $r_c = \beta/\alpha$, which is the slope of the preferred longitude $\lambda$ of $c \subset S^3$ with respect to $(s, t)$–basis.

Claim 3.1 $M_c(K, m)$ is a (possibly degenerate) Seifert fiber space $S^2(b, r_1, r_2, r_c)$; if $r_c = -1/0$, then it is a connected sum of two lens spaces.

Proof of Claim 3.1. $M_c(K, m)$ is regarded as a 3–manifold obtained from $K(m)$ by performing $\lambda$–surgery along the fiber $c \subset K(m)$. Since $[\lambda] = -\alpha[s] - \beta[t]$, $M_c(K, m)$ is a (possibly degenerate) Seifert fiber space $S^2(b, r_1, r_2, r_c)$. If $\alpha = 0$, i.e. $r_c = -1/0$, then $M_c(K, m)$ has a degenerate Seifert fibration and it is a connected sum of two lens spaces.

Recall that $(K_n, m_n)$ is a Seifert surgery obtained from $(K, m)$ by twisting $n$ times along $c$. The image of $c$ after the $n$–twist along $c$ is also a seiferter for $(K_n, m_n)$ and denoted by $e_n$. We study how the Seifert invariant of $K(m)$ behaves under the twisting. We compute the Seifert invariant of $e_n$ in $K_n(m_n)$ under the same cross section on $E$.

Since we have

$$
\begin{pmatrix}
[\mu] \\
[\lambda]
\end{pmatrix} =
\begin{pmatrix}
\alpha_3 & \beta_3 \\
-\alpha & -\beta
\end{pmatrix}
\begin{pmatrix}
[s] \\
[t]
\end{pmatrix},
$$

it follows that

$$
\begin{pmatrix}
[s] \\
[t]
\end{pmatrix} =
\begin{pmatrix}
-\beta & -\beta_3 \\
\alpha & \alpha_3
\end{pmatrix}
\begin{pmatrix}
[\mu] \\
[\lambda]
\end{pmatrix}.
$$

Twisting $n$ times along $c$ is equivalent to performing $-1/n$–surgery on $c$. A preferred meridian-longitude pair $(\mu_n, \lambda_n)$ of $N(c_n) \subset S^3$ satisfies $[\mu_n] = [\mu] - n[\lambda]$ and $[\lambda_n] = [\lambda]$ in $H_1(\partial N(c_n)) = H_1(\partial N(c))$. 

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Kimihiko Motegi
We thus have
\[
\begin{pmatrix}
[s] \\
[t]
\end{pmatrix} = \begin{pmatrix}
-\beta & -n\beta - \beta_3 \\
\alpha & n\alpha + \alpha_3
\end{pmatrix} \begin{pmatrix}
[\mu_n] \\
[\lambda_n]
\end{pmatrix},
\]
and it follows that
\[
\begin{pmatrix}
[\mu_n] \\
[\lambda_n]
\end{pmatrix} = \begin{pmatrix}
n\alpha + \alpha_3 & n\beta + \beta_3 \\
-\alpha & -\beta
\end{pmatrix} \begin{pmatrix}
[s] \\
[t]
\end{pmatrix}.
\]
Hence, the Seifert invariant of the fiber $c_n$ in $K_n(m_n)$ is $(n\beta + \beta_3)/(n\alpha + \alpha_3)$, and $K_n(m_n) = S^2(b_1, r_1, r_2, (n\beta + \beta_3)/(n\alpha + \alpha_3))$.

**Remark 3.2** Since $(n\beta + \beta_3)/(n\alpha + \alpha_3)$ converges to $\beta/\alpha$ when $|n|$ tends to $\infty$, $M_c(K, m)$ can be regarded as the limit of $K_n(m_n)$ when $|n|$ tends to $\infty$.

We divide into three cases: $r_c = -1/0$, $r_c \in \mathbb{Z}$ or $r_c \in \mathbb{Q} \setminus \mathbb{Z}$. Except for the last case, we do not need the assumption that $M_c(K, m)$ is an L-space.

**Case 1.** $r_c = \beta/\alpha = -1/0$.

Since $\alpha_3 > 0$ and $\alpha_3 - \beta_3 = 1$, we have $\alpha_3 = 1$, $\beta = -1$. Hence $K_n(m_n)$ is a Seifert fiber space $S^2(b, r_1, r_2, (n\beta + \beta_3)/(n\alpha + \alpha_3)) = S^2(b, r_1, r_2, -n + \beta_3)$, which is a lens space for any $n \in \mathbb{Z}$. Following Claim 2.5 $S^2(b, r_1, r_2, -n + \beta_3)$ is an L-space if $n \neq b + \beta_3 + r_1 + r_2$. Thus $(K_n, m_n)$ is an L-space surgery for all $n \in \mathbb{Z}$ except at most $n = b + \beta_3 + r_1 + r_2$.

Next suppose that $r_c = \beta/\alpha \neq -1/0$. Then the Seifert invariant of $c_n$ is
\[
f(n) = \frac{n\beta + \beta_3}{n\alpha + \alpha_3} = \frac{\beta_3 - \beta}{n\alpha + \alpha_3} = r_c + \frac{\beta_3 - r_c \alpha_3}{n\alpha + \alpha_3}.
\]
Since $\alpha \beta_3 - \beta_3 = \alpha (\beta_3 - r_c \alpha_3) = 1$, $\alpha$ and $\beta_3 - r_c \alpha_3$ have the same sign.

**Case 2.** $r_c \in \mathbb{Z}$. We put $r_c = p$. Then we can write $S^2(b, r_1, r_2, r_c) = S^2(b + p, r_1, r_2)$.

(i) If $b \leq -p - 3$ or $b \geq -p + 1$, then Proposition 2.4(1) shows that $S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p)$ is an L-space if $-1 \leq f(n) - p \leq 1$, i.e. $p - 1 \leq f(n) \leq p + 1$. 

Hence \((K_n, m_n)\) is an L-space for all \(n\) but \(n \in (x_1, x_2)\), where \(f(x_1) = p - 1\) and \(f(x_2) = p + 1\); see Figure 3.1(i).

(ii) If \(b = -p - 2\), then it follows from Proposition 2.4(2), there is an \(\varepsilon > 0\) such that \(S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) = S^2(-2, r_1, r_2, f(n) - p)\) is an L-space if \(1 \leq f(n) - p \leq \varepsilon\). Hence \((K_n, m_n)\) is an L-space except for only finitely many \(n \in (x_1, x_2)\), where \(f(x_1) = p - 1, f(x_2) = p + \varepsilon\); see Figure 3.1(ii).

(iii) Suppose that \(b = -p - 1\). If \(r_1 + r_2 \geq 1\) (resp. \(r_1 + r_2 \leq 1\)), then Proposition 2.4(3) shows that \(S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) = S^2(-1, r_1, r_2, f(n) - p)\) is an L-space if \(0 < f(n) - p \leq 1\) (resp. \(-1 \leq f(n) - p < 0\)). Hence \((K_n, m_n)\) is an L-space for any integer \(n \geq x_2\), where \(f(x_2) = p + 1\) (resp. \(n \leq x_1\), where \(f(x_1) = p - 1\)), see Figure 3.2.

(iv) If \(b = -p\), then Proposition 2.4(4) shows that \(S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) = S^2(r_1, r_2, f(n) - p)\) is an L-space if \(-\varepsilon \leq f(n) - p \leq 1\), i.e.
\( p - \varepsilon \leq f(n) \leq p + 1 \) for some \( \varepsilon > 0 \). Hence \( (K_n, m_n) \) is an L-space for all \( n \) but \( n \in (x_1, x_2) \), where \( f(x_1) = p - \varepsilon \) and \( f(x_2) = p + 1 \); see Figure 3.1(iii).

**Case 3.** \( r_c \in \mathbb{Q} \setminus \mathbb{Z} \) and \( M_c(K, m) = S^2(b, r_1, r_2, r_c) \) is an L-space. We assume \( p < r_c < p + 1 \) for some integer \( p \). Then we have \( S^2(b, r_1, r_2, r_c) = S^2(b + p, r_1, r_2, r_c - p) \), where \( 0 < r_c - p < 1 \).

(i) If \( b \leq -p - 3 \) or \( b \geq -p + 1 \), then Proposition 2.4(1) shows that \( S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) \) is an L-space if \(-1 \leq f(n) - p \leq 1\), i.e. \( p - 1 \leq f(n) \leq p + 1 \). Hence \( (K_n, m_n) \) is an L-space for all \( n \) but \( n \in (x_1, x_2) \), where \( f(x_1) = p - 1 \) and \( f(x_2) = p + 1 \); see Figure 3.3(i).

(ii) Suppose that \( b = -p - 1 \). Since \( S^2(b, r_1, r_2, r_c) = S^2(b + p, r_1, r_2, r_c - p) = S^2(-1, r_1, r_2, r_c - p) \) is an L-space, by Proposition 2.6(1), \( S^2(b, r_1, r_2, f(n)) = S^2(-1, r_1, r_2, f(n) - p) \) is an L-space if \( r_c - p \leq f(n) - p \leq 1 \) (i.e. \( r_c \leq f(n) \leq p + 1 \)). Hence \( (K_n, m_n) \) is an L-space for any \( n \geq x_2 \), where \( f(x_2) = p + 1 \); see Figure 3.3(ii). (Furthermore, if \( r_1 + r_2 \geq 1 \), then by Proposition 2.4(3)(i), \( S^2(-1, r_1, r_2, f(n) - p) \) is an L-space provided \( 0 \leq f(n) - p \leq 1 \), i.e. \( p < f(n) \leq p + 1 \). Hence \( (K_n, m_n) \) is an L-space surgery for any integer \( n \) except for \( n \in [x_1, x_2) \), where \( f(x_1) = p \) and \( f(x_2) = p + 1 \). )

![Figure 3.3: f(x) = \( \frac{\beta_3 + \beta_5}{\alpha + \alpha_3} \)](image)

(iii) Suppose that \( b = -p - 2 \). Since \( S^2(b, r_1, r_2, r_c) = S^2(b + p, r_1, r_2, r_c - p) = S^2(-2, r_1, r_2, r_c - p) \) is an L-space, following Proposition 2.6(2), \( S^2(b, r_1, r_2, f(n)) = S^2(-2, r_1, r_2, f(n) - p) \) is an L-space if \(-1 \leq f(n) - p \leq r_c - p \) (i.e. \( p - 1 \leq f(n) \leq r_c \)). Hence \( (K_n, m_n) \) is an L-space for any \( n \leq x_1 \), where \( f(x_1) = p - 1 \); see Figure 3.4(i). (Furthermore, if \( r_1 + r_2 \leq 1 \), then Proposition 2.4(2) shows that \( S^2(-2, r_1, r_2, f(n) - p) \) is an L-space provided \(-1 \leq f(n) - p < 1 \), i.e. \( p - 1 \leq f(n) < p + 1 \). Hence \( (K_n, m_n) \)
is an L-space surgery for any integer \( n \) except for \( n \in (x_1, x_2) \), where \( f(x_1) = p - 1 \) and \( f(x_2) = p + 1 \).

(iv) If \( b = -p \), then Proposition 2.4(4) shows that \( S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) = S^2(r_1, r_2, f(n) - p) \) is an L-space if \( -\varepsilon \leq f(n) - p \leq 1 \), i.e. \( p - \varepsilon \leq f(n) \leq p + 1 \) for some \( \varepsilon > 0 \). Hence \( (K_n, m_n) \) is an L-space for all \( n \) but \( n \in (x_1, x_2) \), where \( f(x_1) = p - \varepsilon \) and \( f(x_2) = p + 1 \); see Figure 3.4(ii).

Figure 3.4: \( f(x) = \frac{\beta_2 + \beta_3}{\alpha_1 + \alpha_3} \)

Now let us prove the “only if” part of Theorem 1.4. We begin by observing:

**Lemma 3.3** \( M_c(K, m) \) cannot be \( S^2 \times S^1 \), in particular, if \( M_c(K, m) \) is a lens space, then it is an L-space.

**Proof of Lemma 3.3.** Let \( w \) be the linking number between \( c \) and \( K \). Then \( H_1(M_c(K, m)) = \langle \mu_c, \mu_K \mid w\mu_c + m\mu_K = 0, w\mu_K = 0 \rangle \), where \( \mu_c \) is a meridian of \( c \) and \( \mu_K \) is that of \( K \). If \( M_c(K, m) \cong S^2 \times S^1 \), then \( H_1(M_c(K, m)) \cong \mathbb{Z} \), and we have \( w = 0 \). Let us put \( V = S^3 - \text{int}N(c) \), which is a solid torus containing \( K \) in its interior; \( K \) is not contained in any 3-ball in \( V \). Since \( w = 0 \), \( K \) is null-homologous in \( V \). Furthermore, since \( c \) is a seiferter for \( (K, m) \), the result \( V(K; m) \) of \( V \) after \( m \)-surgery on \( K \) has a (possibly degenerate) Seifert fibration. Then [13, Lemma 3.22] shows that the Seifert fibration of \( V(K; m) \) is non-degenerate and neither a meridian nor a longitude of \( V \) is a fiber in \( V(K; m) \), and the base surface of \( V(K; m) \) is not a Möbius band. Since \( K \) is null-homologous in \( V \), \( V(K; m) \) is not a solid torus [18, Theorem 1.1], and hence, \( V(K; m) \) has a Seifert fibration over the disk with at least two exceptional fibers. Then \( M_c(K, m) = V(K; m) \cup N(c) \) is obtained by attaching \( N(c) \) to \( V(K; m) \) so that the meridian of \( N(c) \) is identified with a meridian of \( V \). Since a regular fiber on \( \partial V(K; m) \) intersects a meridian of \( V \), i.e. a meridian of \( N(c) \) more than
once, \( M_c(K, m) \) is a Seifert fiber space over \( S^2 \) with at least three exceptional fibers. Therefore \( M_c(K, m) \) cannot be \( S^2 \times S^1 \). This completes a proof. ∎(Lemma 3.3)

Suppose first that \( K(m) \) is a lens space and \( c \) is a core of a genus one Heegaard splitting of \( K(m) \). Then \( V(K; m) = K(m) - \text{int}(c) \) is a solid torus and \( M_c(K, m) = V(K; m) \cup N(c) \) is obviously a lens space. By Lemma 3.3 \( M_c(K, m) \) is an L-space.

In the remaining case, as in the proof of the “if” part of Theorem 1.4, \( M_c(K, m) \) has a form \( S^2(b, r_1, r_2, r_c) \) (0 < \( r_1 \leq r_2 < 1 \)).

Claim 3.4 If \( r_c = -1/0 \) or \( r_c \in \mathbb{Z} \), then \( M_c(K, m) \) is an L-space.

Proof of Claim 3.4. If \( r_c = -1/0 \), then \( M_c(K, m) = S^2(b, r_1, r_2, -1/0) \) is a connected sum of two lens spaces. Since a connected sum of L-spaces is also an L-space [50, 8.1(5)] ([45]), \( M_c(K, m) \) is an L-space. If \( r_c \in \mathbb{Z} \), then \( M_c(K, m) \) is a lens space, hence it is an L-space by Lemma 3.3. ∎(Claim 3.4)

Now suppose that \( M_c(K, m) \) is not an L-space. Then by Claim 3.4 \( r_c \in \mathbb{Q} \setminus \mathbb{Z} \). We write \( r_c = r_c' + p \) so that 0 < \( r_c' < 1 \) and \( p \in \mathbb{Z} \). Then \( M_c(K, m) = S^2(b, r_1, r_2, r_c) = S^2(b + p, r_1, r_2, r_c') \). Since \( M_c(K, m) \) is not an L-space, \( b + p = -1 \) or \( -2 \) (Theorem 2.1). It follows from Proposition 2.7 that there is an \( \varepsilon > 0 \) such that \( K_n(m_n) = S^2(b, r_1, r_2, f(n)) = S^2(b + p, r_1, r_2, f(n) - p) = S^2(-1, r_1, r_2, f(n) - p) \) (resp. \( S^2(-2, r_1, r_2, f(n) - p) \)) is not an L-space if 0 < \( f(n) - p < r_c' + \varepsilon \), i.e. \( p < f(n) < r_c + \varepsilon \) (resp. \( r_c' - \varepsilon < f(n) - p < 1 \), i.e. \( r_c - \varepsilon < f(n) < p + 1 \)). Hence there are at most finitely many integers \( n \) such that \( K_n(m_n) \) is an L-space, i.e. (\( K_n, m_n \)) is an L-space surgery. See Figure 3.5.

![Diagram](image)

Figure 3.5: \( f(x) = \frac{\beta x + \beta_2}{\alpha x + \alpha_3} \)

This completes a proof of Theorem 1.4. ∎(Theorem 1.4)
Proof of Theorem 1.5. Note that $K_n(m)$ is a Seifert fiber space which admits a Seifert fibration over $\mathbb{R}P^2$, or $K_n(m)$ has $S^2 \times S^1$ as a connected summand according as $c$ becomes a non-degenerate fiber, or a degenerate fiber in $K_n(m)$, respectively. In the former case, Boyer, Gordon and Watson [8, Proposition 5] prove that $K_n(m)$ is an L-space. In the latter case, $(K_n(m), n) = (O, 0)$ [17, Theorem 8.1], which is not an L-space surgery, but there is at most one such integer $n$ [13, Theorem 5.1]. This completes a proof. □

Example 3.5 Let us consider a three component link $O \cup c_1 \cup c_2$ depicted in Figure 3.6. It is shown in [13, Lemma 9.26] that $c_1, c_2$ become fibers in a Seifert fibration of $O(0)$. Let $A$ be an annulus in $S^3$ cobounded by $c_1$ and $c_2$. Performing $(-l)$–annulus twist along $A$, equivalently performing $(1/l + 3)$–, $(-1/l + 3)$–surgeries on $c_1, c_2$ respectively, we obtain a knot $K_l$ given by Eudave-Muñoz [15]. Then, as shown in [15], $(K_l, 12l^2 - 4l)$ is a Seifert surgery such that $K_l(12l^2 - 4l)$ is a Seifert fiber space over $\mathbb{R}P^2$ with at most two exceptional fibers $c_1, c_2$ of indices $[l], [3l + 1]$ for $l \neq 0$, where we use the same symbol $c_i$ to denote the image of $c_i$ after $(-l)$–annulus twist along $A$. Let $c$ be one of $c_1$ or $c_2$. Then $c$ is a seiferter for $(K_l, 12l^2 - 4l)$. Theorem 1.5 shows that a knot $K_l(n)$ obtained from $K_l$ by $n$–twist along $c$ is an L-space knot for all integers $n$.

![Figure 3.6](image_url)  

Figure 3.6: $c_1$ and $c_2$ become fibers in a Seifert fibration of $O(0)$.

4 L-space surgeries and twisting along seiferters II – degenerate case

In this section we will prove Theorem 1.6.

Proof of Theorem 1.6. Since $K(m)$ has a degenerate Seifert fibration, it is a lens space or a connected sum of two lens spaces [13, Proposition 2.8].
(1) $K(m)$ is a lens space with degenerate Seifert fibration.

Then there are at most two degenerate fibers in $K(m)$ [13, Proposition 2.8]. Assume that there are exactly two degenerate fibers. Then $(K, m) = (O, 0)$ and the exterior of these two degenerate fibers is $S^1 \times S^1 \times [0, 1]$. If $c$ is a non-degenerate fiber, then $K_n(m_n)$ has $S^2 \times S^1$ as a connected summand for all integers $n$, and thus $(K_n, m_n) = (O, 0)$ for all integers $n$ [17, Theorem 8.1]. This contradicts [13, Theorem 5.1]. If $c$ is one of the degenerate fibers, then $(K_n, m_n)$ is a lens space, which is $S^2 \times S^1$ only when $(K_n, m_n) = (O, 0) = (K_0, m_0)$, i.e. $n = 0$ [13, Theorem 5.1]. Thus $(K_n, m_n)$ is an L-space surgery except when $n = 0$.

Suppose that $K(m)$ has exactly one degenerate fiber $t_d$. There are two cases to consider: $K(m) - \text{int}N(t_d)$ is a fibered solid torus or has a non-degenerate Seifert fibration over the Möbius band with no exceptional fiber ([13, Proposition 2.8]). In either case, a meridian of $t_d$ is identified with a regular fiber on $\partial(K(m) - \text{int}N(t_d))$.

Assume that $K(m) - \text{int}N(t_d)$ is a fibered solid torus. Suppose that $c$ is a non-degenerate fiber. If $c$ is a core of the solid torus, then $K(m) - \text{int}N(c)$ is a solid torus and $K_n(m_n)$ is a lens space. Hence $(K_n, m_n)$ is an L-space surgery except when $K_n(m_n) \cong S^2 \times S^1$, i.e. $(K_n, m_n) = (O, 0)$. By [13, Theorem 5.1] there is at most one such integer $n$. If $c$ is not a core in the fibered solid torus $K(m) - \text{int}N(t_d)$, then $K_n(m_n)$ is a lens space ($\not\cong S^2 \times S^1$), a connected sum of two lens spaces, or a connected sum of $S^2 \times S^1$ and a lens space ($\not\cong S^2$, $S^2 \times S^1$). The last case cannot happen for homological reasons, and hence $(K_n, m_n)$ is an L-space surgery. If $c$ is the degenerate fiber $t_d$, then $K_n(m_n)$ is a lens space, and except for at most integer $n_0$ with $(K_{n_0}, m_{n_0}) = (O, 0)$, $(K_n, m_n)$ is an L-space surgery.

Next consider the case where $(K, m) = (O, 0)$; see [13, Proposition 2.8]. If $c$ is a non-degenerate fiber, $K_n(m_n)$ has $S^2 \times S^1$ as a connected summand for all integers $n$. This implies that $(K_n, m_n) = (O, 0)$ for all $n$ [17, Theorem 8.1], contradicting [13, Theorem 5.1]. Thus $c$ is a degenerate fiber, and $K_n(m_n)$ ($n \neq 0$) is a Seifert fiber space over $R\mathbb{P}^2$ with at most one exceptional fiber, which has finite fundamental group. Hence for any non-zero integer $n$, $(K_n, m_n)$ is an L-space [47, Proposition 2.3]. It follows that if $c$ is a fiber in a degenerate Seifert fibration of a lens space $K(m)$, then $(K, m)$ is an L-space surgery except for at most one integer $n$.

(2) $K(m)$ is a connected sum of two lens spaces.

It follows from [13, Proposition 2.8] that $K(m)$ has exactly one degenerate fiber $t_d$ and $K(m) - \text{int}N(t_d)$ is a Seifert fiber space over the disk with two exceptional fibers. Note
that a meridian of $t_d$ is identified with a regular fiber on $\partial(K(m) - \text{int}N(t_d))$. We divide into two cases according as $c$ is a non-degenerate fiber or a degenerate fiber.

(i) $c$ is a non-degenerate fiber.

By [13, Corollary 3.21(1)] $c$ is not a regular fiber. Hence $c$ is an exceptional fiber, and $K_n(m_n)$ is a lens space ($\not\cong S^2 \times S^1$), a connected sum of two lens spaces, or a connected sum of $S^2 \times S^1$ and a lens space ($\not\cong S^3$, $S^2 \times S^1$). The last case cannot happen for homological reasons. Hence $(K_n, m_n)$ is an L-space surgery for any integer $n$.

(ii) $c$ is a degenerate fiber, i.e. $c = t_d$.

As in the proof of Theorem 1.4, let $E$ be $K(m) - \text{int}N(c)$ with a fibered tubular neighborhood of the union of two exceptional fibers $t_1, t_2$ and one regular fiber $t_0$ removed. Then $E$ is a product circle bundle over the fourth punctured sphere. Take a cross section of $E$ such that $K(m)$ has a Seifert invariant $S^2(b, r_1, r_2, 1/0)$, where the Seifert invariant of $t_0$ is $b \in \mathbb{Z}$, that of $t_i$ is $0 < r_i < 1$ (i = 1, 2), and that of $c$ is 1/0. We may assume that $r_1 \leq r_2$. Let $s$ be the boundary curve on $\partial N(c)$ of the cross section so that $[s] \cdot [t] = 1$ for a regular fiber $t \subset \partial N(c)$. Then $[\mu] = [t] \in H_1(\partial N(c))$ and $[\lambda] = -[s] - \beta[t] \in H_1(\partial N(c))$ for some integer $\beta$, i.e. we have:

$$
\begin{pmatrix}
[\mu] \\
[\lambda]
\end{pmatrix}
= 
\begin{pmatrix}
0 & 1 \\
-1 & -\beta
\end{pmatrix}
\begin{pmatrix}
[s] \\
[t]
\end{pmatrix}
$$

Let $c_n$ be the image of $c$ after $n$--twist along $c$. Then the argument in the proof of Theorem 1.4 shows that a preferred meridian-longitude pair $(\mu_n, \lambda_n)$ of $\partial N(c_n)$ has the expression:

$$
\begin{pmatrix}
[\mu_n] \\
[\lambda_n]
\end{pmatrix}
= 
\begin{pmatrix}
 n & n\beta + 1 \\
-1 & -\beta
\end{pmatrix}
\begin{pmatrix}
[s] \\
[t]
\end{pmatrix}
$$

Thus $K_n(m_n) = S^2(b, r_1, r_2, (n\beta + 1)/n) = S^2(b + \beta, r_1, r_2, (n\beta + 1)/n - \beta) = S^2(b + \beta, r_1, r_2, 1/n)$ for non-zero integer $n$.

**Claim 4.1** $K_n(m_n)$ is an L-space for $n = 0, \pm 1$.

**Proof of Claim 4.1.** Recall that $K_0(m_0) = K(m)$ is a connected sum of two lens spaces $L_1$ and $L_2$ such that $H_1(L_1) \cong \mathbb{Z}_{\alpha_1}$ and $H_1(L_2) \cong \mathbb{Z}_{\alpha_2}$, where $r_i = \beta_i/\alpha_i$. Thus $K_0(m_0)$ is an L-space. Since $K_{-1}(m_{-1})$ and $K_1(m_1)$ are lens spaces, it remains to show...
that they are not $S^2 \times S^1$. Assume for a contradiction that $K_1(m_1)$ or $K_{-1}(m_{-1})$ is $S^2 \times S^1$. Then Claim 2.5 shows that $r_1 + r_2 = 1$, hence $r_2 = \beta_2/\alpha_2 = (\alpha_1 - \beta_1)/\alpha_1$. Thus $\alpha_1 = \alpha_2$, and $H_1(K_0(m_0)) \cong \mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2}$ is not cyclic, a contradiction. Hence neither $K_1(m_1)$ nor $K_{-1}(m_{-1})$ is $S^2 \times S^1$ and they are L-spaces. □(Claim 4.1)

(1) If $b + \beta \leq -3$ or $b + \beta \geq 1$, Proposition 2.4(1) shows that $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an L-space if $-1 \leq 1/n \leq 1$, i.e. $n \leq -1$ or $n \geq 1$. See Figure 4.1(i). Since $K_0(m_0)$ is also an L-space (Claim 4.1), $K_n(m_n)$ is an L-space for any integer $n$.

(2) If $b + \beta = -2$, Proposition 2.4(2) shows that there is an $\varepsilon > 0$ such that $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an L-space if $-1 \leq 1/n \leq \varepsilon$. Hence $K_n(m_n)$ is an L-space if $n \leq -1$ or $n \geq 1/\varepsilon$. See Figure 4.1(ii). This, together with Claim 4.1, shows that $K_n(m_n)$ is an L-space if $n \leq 1$ or $n \geq 1/\varepsilon$.

(3) Suppose that $b + \beta = -1$. Then Proposition 2.4(3) shows that if $r_1 + r_2 \geq 1$ (resp. $r_1 + r_2 \leq 1$), $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an L-space for any integer $n$ satisfying $0 < 1/n \leq 1$ (resp. $-1 \leq 1/n < 0$), i.e. $n \geq 1$ (resp. $n \leq -1$). See Figure 4.1(i). Combining Claim 4.1, we see that $K_n(m_n)$ is an L-space for any $n \geq -1$ (resp. $n \leq 1$).

(4) If $b + \beta = 0$, then Proposition 2.4(4) shows that there is an $\varepsilon > 0$ such that $K_n(m_n) = S^2(b + \beta, r_1, r_2, 1/n)$ is an L-space if $-\varepsilon \leq 1/n \leq 1$. Hence $K_n(m_n)$ is an L-space if $n \geq 1$ or $n \leq -1/\varepsilon$. See Figure 4.1(iii). This, together with Claim 4.1, shows that $K_n(m_n)$ is an L-space if $n \geq -1$ or $n \leq -1/\varepsilon$.

This completes a proof of Theorem 1.6. □(Theorem 1.6)

Figure 4.1: (i) $-1 \leq 1/n \leq 1$ if $n \leq -1$ or $n \geq 1$, (ii) $-1 \leq 1/n \leq \varepsilon$ if $n \leq -1$ or $n \geq 1/\varepsilon$, (iii) $-\varepsilon \leq 1/n \leq 1$ if $n \leq -1/\varepsilon$ or $n \geq 1$. 
As shown by Greene [23, Theorem 1.5], if $K(m)$ is a connected sum of lens spaces, then $K$ is a torus knot or a cable of a torus knot. More precisely, $(K, m) = (T_{p, q}, pq)$ or $(C_{p, q}(T_{r, s}), pq)$, where $p = qrs \pm 1$. Note that $T_{p, q}(pq) = L(p, q)\sharp L(q, p)$ and $C_{p, q}(T_{r, s})(pq) = L(p, qs^2)\sharp L(q, \pm 1)$.

Let us continue to prove Theorem 1.7 which is a refinement of Theorem 1.6(2).

**Proof of Theorem 1.7.** In the following $(K, m)$ is either $(T_{p, q}, pq)$ or $(C_{p, q}(T_{r, s}), pq)$, where $p, q \geq 2$ and $p = qrs \pm 1$. If $c$ becomes a non-degenerate fiber in $K(m)$, then as shown in the proof of Theorem 1.6, $K_n$ is an L-space knot for any integer $n$. So we assume that $c$ becomes a degenerate fiber in $K(m)$. Recall from Theorem 3.19(3) in [13] that the linking number $l$ between $c$ and $K$ is not zero. Recall also that $K_n(m_n)$ is expressed as $S^2(b + \beta, r_1, r_2, 1/n) = S^2(b + \beta, \beta_1/\alpha_1, \beta_2/\alpha_2, 1/n)$, where $0 < r_1 = \beta_i/\alpha_i < 1$ and $\alpha_i \geq 2$. See the proof of Theorem 1.6. Note that \{\alpha_1, \alpha_2\} = \{p, q\}, and $\alpha_1\alpha_2 = pq \geq 6$.

**Claim 4.2** \( b + \beta \neq -2 \).

**Proof of Claim 4.2.** Assume for a contradiction that $b + \beta = -2$. Then $K_1(m_1) = S^2(-2, \beta_1/\alpha_1, \beta_2/\alpha_2, 1) = S^2(-1, \beta_1/\alpha_1, \beta_2/\alpha_2)$. Hence $|H_1(K_1(m_1))| = | -\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 |$, which coincides with $pq + \hat{p} = \alpha_1\alpha_2 + \hat{p}$. Since $\alpha_1\alpha_2 + \hat{p} > \alpha_1\alpha_2$, we have $| -\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 | > \alpha_1\alpha_2$. This then implies $\beta_1/\alpha_1 + \beta_2/\alpha_2 > 2$ or $\beta_1/\alpha_1 + \beta_2/\alpha_2 < 0$. Either case cannot happen, because $0 < \beta_i/\alpha_i < 1$. Thus $b + \beta \neq -2$. \(\square\)(Claim 4.2)

**Claim 4.3** If $b + \beta = -1$, $\beta_1/\alpha_1 + \beta_2/\alpha_2 > 1$.

**Proof of Claim 4.3.** If $b + \beta = -1$, then $K_1(m_1) = S^2(-1, \beta_1/\alpha_1, \beta_2/\alpha_2, 1) = S^2(\beta_1/\alpha_1, \beta_2/\alpha_2)$. Thus $|H_1(K_1(m_1))| = \alpha_1\beta_2 + \alpha_2\beta_1$, which coincides with $pq + \hat{p} = \alpha_1\alpha_2 + \hat{p}$. Since $\alpha_1\alpha_2 + \hat{p} > \alpha_1\alpha_2$, we have $\alpha_1\beta_2 + \alpha_2\beta_1 > \alpha_1\alpha_2$. This shows $\beta_1/\alpha_1 + \beta_2/\alpha_2 > 1$. \(\square\)(Claim 4.3)

Claims 4.2 and 4.3, together with the argument in the proof of Theorem 1.6 prove that $K_n$ is an L-space knot for any $n \geq -1$.

Now let us prove that $K_n$ is an L-space knot for all integers $n$ under the assumption $\hat{p} \geq 2pq$.

**Claim 4.4** If $\hat{p} \geq 2pq$, then $b + \beta \neq -1$.
Proof of Claim 4.4. Assume that \( l^2 \geq 2pq \), and suppose for a contradiction \( b + \beta = -1 \). Then \( K_{-1}(m_{-1}) = S^2(-1, \beta_1/\alpha_1, \beta_2/\alpha_2, -1) = S^2(-2, \beta_1/\alpha_1, \beta_2/\alpha_2) \), and \( |H_1(K_{-1}(m_{-1})))| = | -2\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 | \), which coincides with \( |pq - l^2| \). The assumption \( l^2 \geq 2pq = 2\alpha_1\alpha_2 \) implies that \( |pq - l^2| = l^2 - pq = l^2 - \alpha_1\alpha_2 \geq \alpha_1\alpha_2 \). Hence \( -2\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 = |pq - l^2| \geq \alpha_1\alpha_2 \). Thus we have \( \beta_1/\alpha_1 + \beta_2/\alpha_2 \geq 3 \) or \( \beta_1/\alpha_1 + \beta_2/\alpha_2 \leq 1 \). The former case cannot happen because \( 0 < \beta_i/\alpha_i < 1 \), and the latter case contradicts Claim 4.3 which asserts \( \beta_1/\alpha_1 + \beta_2/\alpha_2 > 1 \). Hence \( b + \beta \neq -1 \).

\( \square \) (Claim 4.4)

Claim 4.5 If \( l^2 \geq 2pq \), then \( b + \beta \neq 0 \).

Proof of Claim 4.5. Suppose for a contradiction that \( b + \beta = 0 \). Then \( K_{-1}(m_{-1}) = S^2(0, \beta_1/\alpha_1, \beta_2/\alpha_2, -1) = S^2(-1, \beta_1/\alpha_1, \beta_2/\alpha_2) \), and \( |H_1(K_{-1}(m_{-1})))| = | -\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1 | \), which coincides with \( |pq - l^2| \). Since \( l^2 \geq 2pq = 2\alpha_1\alpha_2 \), \( |pq - l^2| = l^2 - pq = l^2 - \alpha_1\alpha_2 \geq \alpha_1\alpha_2 \). Thus we have \( |\alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_2\beta_1| = |pq - l^2| \geq \alpha_1\alpha_2 \). This then implies \( \beta_1/\alpha_1 + \beta_2/\alpha_2 \geq 2 \) or \( \beta_1/\alpha_1 + \beta_2/\alpha_2 \leq 0 \). Either case cannot happen, because \( 0 < \beta_i/\alpha_i < 1 \). Thus \( b + \beta \neq 0 \).

\( \square \) (Claim 4.5)

Under the assumption \( l^2 \geq 2pq \), Claims 4.2, 4.4 and 4.5 imply that \( b + \beta \leq -3 \) or \( b + \beta \geq 1 \). Then the proof of Theorem 1.6 enables us to conclude that \( K_n \) is an L-space knot for all integers \( n \).

\( \square \) (Theorem 1.7)

Example 4.6 Let \( K \) be a torus knot \( T_{3,2} \) and \( c \) an unknotted circle depicted in Figure 4.2; the linking number between \( c \) and \( T_{3,2} \) is 5. Then \( c \) coincides with \( c_{3,2}^+ \) in Section 5, and it is a seiferter for \( T_{3,2}, 6 \). Let \( K_n \) be a knot obtained from \( T_{3,2} \) by \( n \)-twist along \( c \). Since \( 5^2 > 2 \cdot 3 \cdot 2 = 12 \), following Theorem 1.7 \( K_n \) is an L-space knot for all integers \( n \).

![Figure 4.2: c is a seiferter for \( T_{3,2}, 6 \).](image)

Example 4.7 below gives an example of a seiferter for \( (K, m) \), where \( K \) is a cable of a torus knot and \( K(m) \) is a connected sum of two lens spaces.
Let us take an unknotted circle $c$ such that the linking number between $c$ and $k$ is $4p + 1$ and $(-1)$-twist along $c$ converts $(k, 22p^2 + 9p + 1)$ into $(C_{6p+1,p}(T_{3,2}), p(6p + 1))$. Since $p > 1$, $C_{6p+1,p}(T_{3,2})$ is a nontrivial cable of $T_{3,2}$. Thus $c$ is a seiferter for $(C_{6p+1,p}(T_{3,2}), p(6p + 1))$. Let $K_n$ be a knot obtained from $C_{6p+1,p}(T_{3,2})$ by $n$-twist along $c$ so that $K_1 = k$. Since $(4p + 1)^2 \geq 2(6p + 1)p$, Theorem 1.7 shows that $K_n$ is an L-space knot for all integers $n$.

Finally we show that $K_n$ is hyperbolic if $|n| > 3$. As shown in [11, Figure 41], $K_n$ admits a Seifert surgery yielding a small Seifert space which is not a lens space, so we see that $c$ becomes a degenerate fiber in $C_{6p+1,p}(T_{3,2})(p(6p + 1))$ [13, Lemma 5.6(1)]. Hence Corollary 3.21(3) in [13] shows that the link $C_{6p+1,p}(T_{3,2}) \cup c$ is hyperbolic. Now the result follows from [13, Proposition 5.11(3)].

We close this section with the following observation, which shows the non-uniqueness of degenerate Seifert fibration of a connected sum of two lens spaces.

Let $c$ be a seiferter for $(T_{p,q}, pq)$ which becomes a degenerate fiber in $T_{p,q}(pq)$. As the simplest example of such a seiferter $c$, take a meridian $c_\mu$ of $T_{p,q}$. Then $c_\mu$ is isotopic to the core of the filled solid torus (i.e. the dual knot of $T_{p,q}$) in $T_{p,q}(pq)$, which is a degenerate fiber. Hence $c_\mu$ is a seiferter for $(T_{p,q}, pq)$ which becomes a degenerate fiber in $T_{p,q}(pq)$, and $T_{p,q} - \text{int}(c_\mu)$ is homeomorphic to $S^3 - \text{int}(T_{p,q})$. However, in general, $T_{p,q}(pq) - \text{int}(c)$ is not necessarily homeomorphic to $S^3 - \text{int}(T_{p,q})$.

**Example 4.8** Let us take an unknotted circle $c$ as in Figure 4.3. Then $c$ is a seiferter for $(T_{5,3}, 15)$ which becomes a degenerate fiber in $T_{5,3}(15)$, but $T_{5,3}(15) - \text{int}(c)$ is not homeomorphic to $S^3 - \text{int}(T_{5,3})$.

**Proof of Example 4.8.** As shown in Figure 4.3, $T_{5,3}(15)$ is the two-fold branched cover of $S^3$ branched along $L'$ and $c$ is the preimage of an arc $\tau$. Hence $T_{5,3}(15) - \text{int}(c)$ is a Seifert fiber space $D^2(2/3, -2/5)$. Since $|H_1(S^2(2/3, -2/5, x))| = |4 + 15x|$ cannot be 1 for any integer $x$, the Seifert fiber space $T_{5,3}(15) - \text{int}(c)$ cannot be embedded in $S^3$, and hence it is not homeomorphic to $S^3 - \text{int}(T_{5,3})$. (Note that $c$ coincides with $c_{5,3}$ in Section 5.) □ (Example 4.8)
Figure 4.3: $T_{5,3}(15)$ is the two-fold branched cover of $S^3$ branched along $L'$. 

5 **L-space twisted torus knots**

Each torus knot has obviously an unknotted circle $c$ which satisfies the desired property in Question 1.1.

**Example 5.1** Embed a torus knot $T_{p,q}$ into a genus one Heegaard surface of $S^3$. Then cores of the Heegaard splitting $s_p$ and $s_q$ are seiferters for $(T_{p,q}, m)$ for all integers $m$. We call them basic seiferters for $T_{p,q}$; see Figure 5.1. An $n$–twist along $s_p$ (resp. $s_q$) converts $T_{p,q}$ into a torus knot $T_{p+mq}$ (resp. $T_{p+np}$), and hence $n$–twist along a basic seiferter yields an L-space knot for all $n$.

![Figure 5.1: $s_p$ and $s_q$ are basic seiferters for $(T_{p,q}, m)$.](image)

Twistings along a basic seiferter keep the property of being L-space knots, but produce only torus knots. In the following, we will give another circle $c$ such that twistings $T_{p,q}$ along $c$ produce an infinite family of hyperbolic L-space knots.

**Definition 5.2 (twisted torus knot [10])** Let $\Sigma$ be a genus one Heegaard surface of $S^3$. Let $T_{p,q}$ ($p > q \geq 2$) be a $(p,q)$–torus knot which lies on $\Sigma$. Choose an
unknotted circle \( c \subset S^3 - T_{p,q} \) so that it bounds a disk \( D \) such that \( D \cap \Sigma \) is a single arc intersecting \( T_{p,q} \) in \( r \) (\( 2 \leq r \leq p + q \)) points in the same direction. A twisted torus knot \( K(p,q;r,n) \) is a knot obtained from \( T_{p,q} \) by adding \( n \) full twists along \( c \).

**Remark 5.3** Twisting \( T_{p,q} \) along the basic seiferter \( s_p \) (resp. \( s_q \)) \( n \)-times, we obtain the twisted torus knot \( K(p, q; q, n) \) (resp. \( K(p, q; p, n) \)), which is a torus knot \( T_{p+q,q} \) (resp. \( T_{p,q+p} \)), and hence an L-space knot.

In [53] Vafaee studies twisted torus knots from a viewpoint of knot Floer homology and showed that twisted torus knots \( K(p, q; q, n) \) (resp. \( K(p, q; p, n) \)), where \( p > q \geq 2 \) and \( 0 < r < p \) is an L-space knot if and only if either \( r = p - 1 \) or \( r \in \{2, p - 2\} \) and \( n = 1 \). We will give yet more twisted torus knots which are L-space knots by combining seiferter technology and Theorem 1.7.

**Proof of Theorem 1.8.** In the following, let \( \Sigma \) be a genus one Heegaard surface of \( S^3 \), which bounds solid tori \( V_1 \) and \( V_2 \).

- \( K(p, q; p + q, n) \) (\( p > q \geq 2 \)). Given any torus knot \( T_{p,q} \) (\( p > q \geq 2 \)) on \( \Sigma \), let us take an unknotted circle \( c^+_{p,q} \) in \( S^3 - T_{p,q} \) as depicted in Figure 5.2(i); the linking number between \( c^+_{p,q} \) and \( T_{p,q} \) is \( p + q \).

\[
\begin{aligned}
\Sigma & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qu...
into the twisted torus knot $K(p, q; p + q, n)$. Since $c_{p,q}^{+}$ is a seiferter for $(T_{p,q}, pq)$ and $(p + q)^2 = p^2 + q^2 + 2pq > 2pq$, we can apply Theorem 1.7 to conclude that $T(p, q, p + q, n)$ is an L-space knots for all integers $n$.

We show that $T(p, q, p+q, n)$ is hyperbolic if $|n| > 3$. By linking number consideration, we see that $c_{p,q}^{+}$ is not a basic seiferter. Then Corollary 3.21(3) in [13] ([36, Claim 9.2]) shows that $T_{p,q} \cup c_{p,q}^{+}$ is a hyperbolic link. Thus [13, Proposition 5.11(2)] that $K(p, q; p + q, n)$ is a hyperbolic knot if $|n| > 3$.

- $K(p, q; p - q, n)$ ($p > q \geq 2$). Suppose that $p - q \neq 1$. Then let us take $c_{p,q}^{-}$ as in Figure 5.3(i) instead of $c_{p,q}^{+}$; the linking number between $c_{p,q}^{-}$ and $T_{p,q}$ is $p - q$. It follows from [13, Remark 4.7] that $c_{p,q}^{-}$ is also a seiferter for $(T_{p,q}, pq)$ and the link $T_{p,q} \cup c_{p,q}^{-}$ is hyperbolic. Note that if $p - q = 1$, then $c_{p,q}^{-}$ is a meridian of $T_{p,q}$. As above we see that each arc $c_{p,q}^{-} \cap V_i$ is isotoped in $V_i$ to an arc $\gamma \subset \Sigma$ depicted in Figure 5.3(ii) leaving its endpoints fixed. So a disk $D$ bounded by $c_{p,q}^{-}$ can be isotoped so that $D \cap \Sigma = \gamma$, which intersects $T_{p,q}$ in $p - q$ points in the same direction. Thus $n$–twist along $c_{p,q}^{-}$ converts $T_{p,q}$ into the twisted torus knot $K(p, q; p - q, n)$. Since $c_{p,q}^{-}$ is a seiferter for $(T_{p,q}, pq)$, Theorem 1.7 shows that $T(p, q, p - q, n)$ is an L-space knot for any $n \geq -1$. Following [13, Proposition 5.11(2)] $T(p, q, p - q, n)$ is a hyperbolic knot if $|n| > 3$.

- $K(3p + 1, 2p + 1; 4p + 1, n)$ ($p > 0$). Let $k$ be a torus knot $T_{p,2p+1}$ on a genus two Heegaard surface, with unknotted circles $\alpha$ and $c$ as shown in Figure 5.4. Applying $1$–twist along $\alpha$, we obtain a torus knot $T_{3p+1,2p+1}$. We continue to use the same symbol $c$ to denote the image of $c$ after $1$–twist along $\alpha$; the linking number between $c$ and $T_{3p+1,2p+1}$ is $4p + 1$. Note that $1$–twist along $c$ converts $T_{3p+1,2p+1}$ into a Berge knot $Spor[p]$ as shown in [11, Subsection 8.2]. Following [11, Lemma 8.4] $c$ is a seiferter for a lens space surgery $(Spor[p], 22p^2 + 13p + 2) = (Spor[p], (3p + 1)(2p+1)+(4p+1)^2)$. Thus $c$ is also a seiferter for $(T_{3p+1,2p+1}, (3p+1)(2p+1))$. Let
Let $D$ be a disk bounded by $c$. Then $T_{3p+1,2p+1} \cup D$ can be isotoped so that $T_{3p+1,2p+1}$ lies on $\Sigma$, $D \cap \Sigma$ consists of a single arc, which intersects $T_{3p+1,2p+1}$ in $4p + 1$ points in the same direction. Thus $n$–twist along $c$ converts $T_{3p+1,2p+1}$ into a twisted torus knot $K(3p + 1, 2p + 1; 4p + 1, n)$. Since $c$ is a seiferter for $(T_{3p+1,2p+1}, (3p + 1)(2p + 1) + (4p + 1)^2 > 2(3p + 1)(2p + 1)$, Theorem 1.7 shows that $K(3p + 1, 2p + 1; 4p + 1, n)$ is an L-space knot for all integers $n$. Let us observe that $K(3p + 1, 2p + 1; 4p + 1, n)$ is a hyperbolic knot if $|n| > 3$. Figure 44 in [11] shows that $n$–twist converts $(T_{3p+1,2p+1}, (3p + 1)(2p + 1))$ into a Seifert surgery which is not a lens space surgery if $|n| \geq 2$. Hence $c$ becomes a degenerate fiber in $T_{3p+1,2p+1}(3p + 1)(2p + 1)$ [13, Lemma 5.6(1)], and Corollary 3.21(3) in [13] shows that the link $T_{3p+1,2p+1} \cup c$ is hyperbolic. Now the result follows from [13, Proposition 5.11(2)].

![Figure 5.4: A surgery description of $T_{3p+1,2p+1}$ and a seiferter $c$](image)

- $K(3p + 2, 2p + 1; 4p + 3, n)$ ($p > 0$). As above, we follow the argument in [11, Subsection 8.3], but we need to take the mirror image at the end. Take a torus knot $k = T_{-p-1,2p+1}$ on a genus two Heegaard surface of $S^3$, unknotted circles $\alpha'$ and $c'$ as shown in Figure 5.5. Then $(-1)$–twist along $\alpha'$ converts $T_{-p-1,2p+1}$ into $T_{-3p-2,2p+1}$. As above we denote the image of $c'$ after $(-1)$–twist along $\alpha'$ by the same symbol $c'$; the linking number between $c'$ and $T_{-3p-2,2p+1}$ is $4p + 3$. Note that $(-1)$–twist along $c'$ converts $T_{-3p-2,2p+1}$ into a Berge knot $\text{Spore}[p]$ as shown in [11, Subsection 8.3]. Then Lemma 8.6 in [11] shows that $c'$ is a seiferter for a lens space surgery $(\text{Spore}[p], -22p^2 - 31p - 11) = (\text{Spore}[p], (-3p - 2)(2p + 1) - (4p + 3)^2)$. Thus $c'$ is also a seiferter for $(T_{-3p-2,2p+1}, (-3p - 2)(2p + 1))$. Let $D'$ be a disk bounded by $c'$. Then $T_{3p+2,2p+1} \cup D'$ can be isotoped so that $T_{3p+2,2p+1}$ lies on $\Sigma$. $D' \cap \Sigma$ consists of a single arc, which intersects $T_{3p+2,2p+1}$ in $4p + 3$ points in the same direction. Now taking the mirror image of $T_{3p+2,2p+1} \cup D'$, we obtain $T_{3p+2,2p+1} \cup D$ with $\partial D = c$; $D \cap \Sigma$ consists of a single arc, and $D$ intersects $T_{3p+2,2p+1}$ in $4p + 3$ points.
in the same direction. Then \( c \) is a seiferter for \((T_{3p+2,2p+1},(3p+2)(2p+1))\). Since 
\((4p+3)^2 > 2(3p+2)(2p+1)\), Theorem 1.7 shows that \( K(3p+2, 2p+1; 4p+3,n) \) is an L-space knot for all integers \( n \). 
Let us show that \( K(3p+2, 2p+1; 4p+3,n) \) is hyperbolic if \( |n| > 3 \). Figure 47 in [11], together with [13, Lemma 5.6(1)], shows that \( c' \) becomes a degenerate fiber in \( T_{-3p-2,2p+1}((-3p-2)(2p+1)) \), and so \( c \) becomes a degenerate fiber in \( T_{3p+2,2p+1}((3p+2)(2p+1)) \). Apply the same argument as above to obtain the desired result.

![Figure 5.5: A surgery description of \( T_{-3p-2,2p+1} \) and a seiferter \( c' \)](image)

- \( K(2p + 3, 2p + 1; 2p + 2, n) \) \((p > 0)\). We follow the argument in [11, Section 6]; as above we need to take the mirror image at the end. Take a torus knot \( k = T_{-3p-2,3} \) on a genus two Heegaard surface of \( S^3 \), unknotted circles \( \alpha' \) and \( \alpha'' \) as in Figure 5.6(i). Then \((-2)\)-twist along \( \alpha'' \) converts the torus knot \( T_{-3p-2,3} \) into a Berge knot \( VI[p] \). Lemma 6.1 in [11] shows that \( c' \), the image of \( c' \) after the \((-2)\)-twist along \( \alpha' \), is a seiferter for a lens space surgery \( (VI[p], -8p^2 - 16p - 7) \); the linking number between \( \alpha'' \) and \( VI[p] \) is \( 2p + 2 \). Now we show that \( \alpha'' \)-twist along \( \alpha'' \) converts \( (VI[p], -8p^2 - 16p - 7) \) into \( (T_{-2p-1,2p+3},(-2p-1)(2p+3)) \). Note that \( c' \) remains a seiferter for \( (T_{-2p-1,2p+3},(-2p-1)(2p+3)) \). Since the linking number between \( \alpha'' \) and \( VI[p] \) is \( 2p + 2 \), the surgery slope \( -8p^2 - 16p - 7 \) becomes \( -8p^2 - 16p - 7 + (2p + 2)^2 = (-2p - 1)(2p + 3) \). Let us observe that the knot obtained from \( VI[p] \) by \( 1 \)-twist along \( \alpha'' \), which has a surgery description given by Figure 5.6(i), is \( T_{-2p-1,2p+3} \). The surgeries described in Figure 5.6(i) can be realized by the following two successive twistings: \( 1 \)-twist along an annulus cobounded by \( c' \) and \( \alpha' \) (cf.[13, Definition 2.32]), and \((-1)\)-twist along \( \alpha' \). The annulus twist converts \( k = T_{-3p-2,3} \) into \( k' = T_{-2p-1,2} \) as shown in Figure 5.6(ii). Then \((-1)\)-twist along \( \alpha' \) changes \( k' = T_{-2p-1,2} \) into \( T_{-2p-1,2p+3} \), which lies on the genus one Heegaard surface \( \Sigma \). Let \( D' \) be a disk bounded by \( c' \). Then \( D' \) can be slightly isotoped so that \( D' \cap \Sigma \) consists of a single arc, which intersects \( T_{-2p-1,2p+3} \) in \( 2p + 2 \) points in the
same direction; see Figure 5.6(ii). Now taking the mirror image of $T_{-2p-1,2p+3} \cup D'$, we obtain $T_{2p+1,2p+3} \cup D$ with $\partial D = c$; $D \cap \Sigma$ consists of a single arc, which intersects $T_{2p+1,2p+3}$ in $2p + 2$ points in the same direction. Then $c$ is a seiferter for $(T_{2p+1,2p+3}, (2p + 1)(2p + 3)) = (T_{2p+3,2p+1}, (2p + 3)(2p + 1))$. Theorem 1.7 shows that $K(2p + 3, 2p + 1; 2p + 2, n)$ is an L-space knot for any integer $n \geq -1$. The hyperbolicity of knots $K(2p + 3, 2p + 1; 2p + 2, n)$ for $|n| > 3$ follows from the same argument as above, in which we refer to Figure 33 instead of Figure 47 in [11].

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.6.png}
\caption{Surgery descriptions of $T_{-2p-1,2p+3}$ and a seiferter $c'$}
\end{figure}

\begin{proof}[Theorem 1.8] Given any torus knot $T_{p,q}$ ($p > q \geq 2$), let us take an unknotted circle $c = c_{p,q}^+$ in $S^3 - T_{p,q}$ (Figure 5.2(i)). Then as shown in the proof of Theorem 1.8, $n$–twist along $c$ converts $T_{p,q}$ into the twisted torus knot $K(p, q; p + q, n)$, which is an L-space knot for all integers $n$ and hyperbolic if $|n| > 3$.

The last assertion of Corollary 1.9 follows from Claim 5.4 below. Thus the unknotted circle $c$ satisfies the required property in Corollary 1.9. 
\end{proof}

\begin{claim} \label{claim5.4}
\{K(p, q; p + q, n)\}_{|n|>3} \text{ is a set of mutually distinct hyperbolic knots.}
\end{claim}

\begin{proof}[Claim 5.4] Recall that $c_{p,q}^+$ is a seiferter for $(T_{p,q}, pq)$ and the linking number between $c_{p,q}^+$ and $T_{p,q}$ is $p + q$. Thus $n$–twist along $c_{p,q}^+$ changes $(T_{p,q}, pq)$ to a Seifert surgery $(K(p, q; p + q, n), pq + n(p + q)^2)$. Note that $K(p, q; p + q, n)(pq + n(p + q)^2)$ is a Seifert fiber space over $S^2$ with at most three exceptional fibers of indices $p$, $q$ and $|n|$. see the proof of Theorem 1.8.

Assume that $K(p, q; p + q, n)$ is isotopic to $K(p, q; p + q, n')$ for some integers $n, n'$ with $|n|, |n'| > 3$. Then $pq + n(p + q)^2$, and $pq + n'(p + q)^2$--surgeries on the hyperbolic knot $K(p, q; p + q, n)$ yield Seifert fiber spaces. Hence $|pq + n(p + q)^2 - (pq + n'(p + q)^2)| =
\end{proof}
\(|n - n'(p + q)^2| \leq 8\) by [32, Theorem 1.2]. Since \(p + q \geq 5\), we have \(n = n'\). This completes a proof. (In the above argument, we can apply [1, Theorem 8.1] which gives the bound 10 instead of 8.) □ (Claim 5.4)

6 L-space twisted Berge knots

In this section we prove Theorem 1.11 using Theorem 1.7 and observations in [13, 11].

Berge [6] gave twelve infinite families of knots which admit lens space surgeries. These knots are referred to as Berge knots of types (I)–(XII) and conjectured to comprise all knots with lens space surgeries. Recall that a Berge knot of type (I) is a torus knot and that of (II) is a cable of a torus knot, henceforth we consider Berge knots of types (III)–(XII).

• Berge knots of types (III)–(VI).

Suppose that \(K\) is a Berge knot of type (III), (IV), (V) or (VI). Then we have an unknotted solid torus \(V\) containing \(K\) in its interior such that \(V(K; m)\) is a solid torus [6, 11], and hence the core \(c\) of the solid torus \(W = S^3 - \text{int}V\) is a seiferter for \((K, m)\) and \((K_n, m_n)\) is also a lens space. If \(K_0(m_n)\) is not an L-space, then it is \(S^2 \times S^1\) and \((K_n, m_n) = (O, 0)\) ([17, Theorem 8.1]). Now let us exclude this possibility. First we note that \(V(K_n, m_n) \cong V(K; m)\) for all integers \(n\) and \(H_1(V(K_n; m_n)) \cong \mathbb{Z} \oplus \mathbb{Z}(m_n, \omega)\) [20, Lemma 3.3], where \(\omega\) is the winding number of \(K\) in \(V\), i.e. the linking number between \(K_n\) and \(c\). Since \(V(K_n; m_n) \cong S^1 \times D^2\), \(K_n\) is a 0 or 1–bridge braid in \(V\) [18], hence \(\omega \geq 2\). This then implies that \(m_n \neq 0\). Hence \((K_n, m_n)\) is an L-space knot for all integers \(n\).

• Berge knots of types (VII), (VIII).

Let \(g_1\) and \(g_2\) be simple closed curves embedded in a genus two Heegaard surface \(F\) of \(S^3\) and \(c\) an unknot in \(S^3\) as in Figure 6.1.

Take a regular neighborhood \(N(g_1 \cup g_2)\) of \(g_1 \cup g_2\) in \(F\), which is a once punctured torus. Then the curve \(\partial N(g_1 \cup g_2)\) becomes a trefoil knot after \((-1)\)–twist along \(c\), and the figure-eight knot after 1–twist along \(c\). Let \(k\) be a knot in \(N(g_1 \cup g_2)\) representing \(a[g_1] + b[g_2] \in H_1(N(g_1 \cup g_2))\), where \(a\) and \(b\) are coprime integers. The we see that \(k\) is a torus knot \(T_{a+b, -a}\). The Berge knot \(K\) of type (VII) (resp. (VIII)) is obtained from \(T_{a+b, -a}\) by \((-1)\)–twist (resp. 1–twist) along \(c\). As shown in
Figure 6.1: A regular neighborhood $N(g_1 \cup g_2)$ of $g_1 \cup g_2$ in $F$ and an unknotted circle $c$

[13, Lemma 4.6], $T_{a+b,-a} \cup c$ is isotopic to $T_{a+b,-a} \cup c_{a+b,-a}^+$ and Berge knot of type (VII) is $K(a + b, -a; |b|, -1)$, that of type (VIII) is $K(a + b, -a; |b|, 1)$; see the proof of Theorem 1.8. (Here we extend the notation $K(p, q; r, n)$ for twisted torus knots in an obvious fashion to include the case where $p, q$ are possibly negative integers.) We assume $|a|, |b| \geq 2$, for otherwise $K(a + b, -a; |b|, \pm 1)$ is a torus knot. Furthermore, if $|a + b| = 1$, then $T_{a+b,-a} \cup c = T_{\pm 1,-a} \cup c$ is a torus link $T_{2,2b}$ or $T_{2,-2b}$, and $K(a + b, -a; |b|, \pm 1)$ is a torus knot, so we assume $|a + b| > 1$. Let $K_n$ be a knot obtained from the Berge knot $K$ by $n$–twist along $c$, i.e. $K_n = K(a + b, -a; |b|, n + \varepsilon)$; $\varepsilon = -1$ if $K$ is of type (VII), $\varepsilon = 1$ if $K$ is of type (VIII). If $a(a + b) < 0$ (i.e. $-a(a + b) > 0$), then by Theorem 1.8 $K_n$ is an L-space knot for any integer $n$. If $a(a + b) > 0$ (i.e. $-a(a + b) < 0$), Theorem 1.8 shows that the mirror image $K(a + b, a; |b|, -n - \varepsilon)$ of $K_n$ is an L-space knot if $-n - \varepsilon \geq -1$, i.e. $n \leq 1 - \varepsilon$. Hence $K_n$ is an L-space knot for any integer $n \leq 1 - \varepsilon$.

- Berge knots of types (IX)–(XII).

These knots are often called sporadic knots and we denote them by $\text{Spora}[p]$, $\text{Sporb}[p]$, $\text{Sporc}[p]$ and $\text{Spord}[p]$ ($p \geq 0$), respectively. It is easy to see that $\text{Spora}[0]$ and $\text{Sporb}[0]$ are trivial knots, $\text{Sporc}[0] = T_{-3,4}$ and $\text{Spord}[0] = T_{-5,3}$. Thus we may assume $p > 0$ for $\text{Sporc}[p]$ ($\chi = a, b, c, d$). Furthermore, we observe that $\text{Spora}[1]$ is obtained from $T_{3,2}$ by 1–twist along the seiferter $c = c_{3,2}^+$; see Figure 4.2. Hence following Example 4.6 a knot $K_n$ obtained from $\text{Spora}[1]$ by $n$–twist along $c$ is an L-space knot for any integer $n$. Thus we may assume $p > 1$ for $\text{Spora}[p]$.

As shown in Example 4.7 the lens space surgery ($\text{Spora}[p], 22p^2 + 9p + 1$) is obtained from $(C_0p+1,p(T_{3,2}), p(6p + 1))$ by 1–twist along the seiferter $c$, and $n$–twist along $c$ converts $C_0p+1,p(T_{3,2})$ into an L-space knot for all integers $n$. Hence $n$–twist changes $\text{Spora}[p]$ to an L-space knot for all integers $n$. 

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\[\text{Kimihiko Motegi}\]
The proof of Theorem 1.8 shows that the lens space surgery \( (\text{Sporb}[p], 22p^2 + 13p + 2) \) is obtained from \((T_{3p+1,2p+1}, (3p+1)(2p + 1))\) by \(1\)–twist along the seiferter \(c\), hence \(K_n\) obtained from \(\text{Sporb}[p]\) by \(n\)–twist along \(c\) is \(K(3p + 1, 2p + 1; 4p + 1, n + 1)\). By Theorem 1.8 \(K_n\) is an L-space knot for all integers \(n\). Similarly \((\text{Sporc}[p], -22p^2 - 31p - 11)\) is obtained from \((T_{-3p-2,2p+1}, (-3p-2)(2p + 1))\) by \((-1)\)–twist along \(c'\), and \(K_n\) obtained from \(\text{Sporc}[p]\) by \(n\)–twist along \(c'\) is \(K(-3p-2, 2p + 1; 4p + 4, n - 1)\), which is the mirror image of \(K(3p + 2, 2p + 1; 4p + 4, -n + 1)\). Theorem 1.8 shows that \(K(3p + 2, 2p + 1; 4p + 4, -n + 1)\) is an L-space knot for any integer \(n\), and thus \(K_n\) is an L-space knot for all integers \(n\).

Finally, let us consider a Berge knot \(\text{Sporc}[p]\) \((p \geq 0)\). Proposition 8.8 in [11] shows that the lens space surgery \((\text{Sporc}[p], -22p^2 - 35p - 14)\) has a seiferter \(c'\) such that the linking number between \(c'\) and \(\text{Sporc}[p]\) is \(4p + 3\) and \(1\)–twist along \(c'\) converts \((\text{Sporc}[p], -22p^2 - 35p - 14)\) into \((C_{-6p-5,p+1}(T_{-3,2}), (-6p-5)(p+1))\) for which \(c'\) is a seiferter. Let \(K_n\) be a knot obtained from \(\text{Sporc}[p]\) by \(n\)–twist along \(c'\), i.e. obtained from \(C_{-6p-5,p+1}(T_{-3,2})\) by \((n - 1)\)–twist along \(c'\). Now we take the mirror image of \(C_{-6p-5,p+1}(T_{-3,2}) \cup c'\) to obtain a link \(C_{6p+5,p+1}(T_{3,2}) \cup c\). Then \(c\) is a seiferter for \((C_{6p+5,p+1}(T_{3,2}), (6p+5)(p+1))\) and \(K_n\) is the mirror image of the knot obtained from \(C_{6p+5,p+1}(T_{3,2})\) by \((-n)\)–twist along \(c\). Since \((4p + 3)^2 \geq 2(6p + 5)(p + 1)\), Theorem 1.7 shows that \(K_n\) is an L-space knot for all integers \(n\).

Let us show that \(K_n\) is a hyperbolic knot except for at most four integers \(n\). Following [13, Theorem 5.10] it is sufficient to observe that \(K \cup c\) is a hyperbolic link. Suppose that \(K\) is a Berge knot of type (III), (IV), (V) or (VI). Then as mentioned above, \(V(K; m)\) is a solid torus, where \(V = S^3 - \text{int}(c)\). By [5, Theorem 3.2] \(V - \text{int}N(K)\) is atoroidal. If \(V - \text{int}N(K)\) is not hyperbolic, then it is Seifert fibered and \(K\) is a torus knot; see [13, Lemma 3.3]. This contradicts the assumption. Hence \(K \cup c\) is a hyperbolic link.

If \(K\) is of type (VII) or (VIII), then \(K \cup c \cong T_{a+b,-a} \cup c_{a+b,-a}^+\) is a hyperbolic link; see the proof of Theorem 1.8.

Assume that \(K\) is of type (IX), i.e. \(K = \text{Sporc}[p]\). Then as shown in the proof of Example 4.7 \(K \cup c\) is a hyperbolic link. In the case where \(K\) is of type (X) or (XI), i.e. \(K = \text{Sporb}[p]\) or \(\text{Sporc}[p]\), then it follows from the proof of Theorem 1.8 that \(K \cup c\) is a hyperbolic link. The argument in the proof of Example 4.7 shows \(K \cup c\) is a hyperbolic link for type (XII) Berge knot \(K = \text{Sporc}[p]\); we refer to Figure 53 instead of Figure 41.

This completes a proof of Theorem 1.11. □(Theorem 1.11)
7 L-space twisted unknots

In [13] we introduced "m–move" to find seiferters for a given Seifert surgery. In particular, m–move is effectively used in [13, Theorem 6.21] to show that \((O, m)\) has infinitely many seiferters for each integer \(m\). Among them there are infinitely many seiferters \(c\) such that \((m, 0)\)–surgery on \(O \cup c\) is an L-space; see Remark 7.3.

Let us take a trivial knot \(c_{m,p}\) in \(S^3 \setminus O\) as illustrated in Figure 7.1, where \(p\) is an odd integer with \(j < p < 3\).

\[
\begin{align*}
\text{Figure 7.1: } O \cup c_{m,p}; \text{ a vertical (resp. horizontal) box with integer } n \text{ denotes a vertical (resp. horizontal) stack of } n \text{ crossings.}
\end{align*}
\]

Then as shown in [13, Theorem 6.21] \(c_{m,p}\) is a seiferter for \((O, m)\) such that \(O \cup c_{m,p}\) is a hyperbolic link in \(S^3\) if \(p \neq 2m + 1\). Denote by \(K_{m,p,n}\) and \(m_{p,n}\) the images of \(O\) and \(m\) after \(n\)–twist along \(c_{m,p}\). Now we investigate \(K_{m,p,n}(m_{p,n})\) using branched coverings and Montesinos trick [38, 39]. Figure 7.2 (b) shows that \(K_{m,p,n}(m_{p,n})\) has an involution with axis L for any integer \(n\). Taking the quotient by this involution, we obtain a 2–fold branched cover \(\pi : K_{m,p,n}(m_{p,n}) \to S^3\) branched along \(L'\) which is the quotient of L; see Figure 7.2(c). As shown in Figure 7.2(d) \(L'\) can be isotoped to a Montesinos link \(M(-n/(mn + 1), (-p + 1)/2p, (-p + 2m + 1)/(-2p + 4m))\). Hence by [38] \(K_{m,p,n}(m_{p,n})\), which is the 2–fold branched cover branched along the Montesinos link \(L'\), is a Seifert fiber space

\[
S^2\left(\frac{-n}{mn + 1}, \frac{-p + 1}{2p}, \frac{p - 2m - 1}{2p - 4m}\right).
\]

The image \(\pi(c_{m,p})\) is an arc \(\tau\) whose ends lie in \(L'\); see Figure 7.2(c) and (d). It follows from [12, Lemma 3.2] that \(c_{m,p}\) is a seiferter for \((K_{m,p,n}, m_{p,n})\); in case of \(n = 0\), \(c_{m,p}\) is a seiferter for \((O, m)\). In the following, the image of \(c_{m,p}\) after \(n\)–twist along itself is denoted by the same symbol.

In what follows assume \(m \leq 0\) and \(p \geq 3\).
Figure 7.2: $K_{m,p,n}(m_p,n)$ is the two-fold branched cover of $S^3$ branched along $L'$.
Proposition 7.1 Assume that \( m \leq 0, \ p \geq 3 \).

1. \((K_{m,p,n}, m_{p,n})\) is an L-space surgery except when \((m,n) = (0,0)\). If \((m,n) = (0,0)\), then \((K_{m,p,n}, m_{p,n}) = (O,0)\) and \(K_{m,p,n}(m_{p,n}) = O(0) \cong S^2 \times S^1\).

2. \(K_{m,p,n}\) is a nontrivial knot if \(n \neq 0\).

3. \(\{K_{m,p,n}\}_{n > 1}\) is a set of mutually distinct hyperbolic L-space knots.

Proof of Proposition 7.1. We note here that the linking number between \(c_{m,p}\) and \(O\) is \(p - m\).

1. Assume first that \(m = 0\). Then \(K_{m,p,n}(m_{p,n})\) is a lens space \(S^2(-n, (-p + 1)/2p, (p - 1)/2p) = S^2(-n - 1, (p + 1)/2p, (p - 1)/2p)\), which is \(S^2 \times S^1\) if and only if \(n = 0\) (Claim 2.5). Hence \(K_{m,p,n}(m_{p,n})\) is an L-space except when \(n = 0\).

Next assume \(m = -1\). Then \(K_{m,p,n}(m_{p,n}) = S^2(-n/(-n + 1), (-p + 1)/2p, (p + 1)/(2p + 4)) = S^2(n/(n - 1), (-p + 1)/2p, (p + 1)/(2p + 4))\). If \(n = 0\) or 2, \(K_{m,p,n}(m_{p,n})\) is a lens space, but it is not \(S^2 \times S^1\), because \(m_{p,n} = -1 + n(m - p)^2 = -1 + n(p + 1)^2 \neq 0\). If \(n = 1\), \(K_{m,p,n}(m_{p,n})\) is a connected sum of two lens spaces, and thus an L-space. Suppose that \(n \neq 0, 1, 2\). In case of \(n < 0\), we have \(0 < n/(n - 1) < 1\) and \(K_{m,p,n}(m_{p,n}) = S^2(n/(n - 1), (-p + 1)/2p, (p + 1)/(2p + 4))\). Note that \((p + 1)/2p + (p + 1)/(2p + 4) = 1/2 + 1/2p + 1/2 - 1/(2p + 4) = 1 + 1/2p - 1/(2p + 4) > 0\). Since \(p \geq 3\), we have \(2p + 4 > 2p > 0\), and hence \(1/2p - 1/(2p + 4) > 0\). It follows that \((p + 1)/2p + (p + 1)/(2p + 4) = 1 + 1/2p - 1/(2p + 4) > 1\). Then Lemma 2.3(2) shows that \(K_{m,p,n}(m_{p,n})\) is an L-space. If \(n < 2\), then \(1 < n/(n - 1) < 2\) and \(K_{m,p,n}(m_{p,n}) = S^2(n/(n - 1), (-p + 1)/2p, (p + 1)/(2p + 4)) = S^2(1/(n - 1), (p + 1)/2p, (p + 1)/(2p + 4))\). Since \(0 < 1/(n - 1), (p + 1)/2p, (p + 3)/2p, (2p + 8)\) are less than 1, \(K_{m,p,n}(m_{p,n})\) is an L-space by Theorem 2.1(1).

Assume that \(m = -2\). Then \(K_{m,p,n}(m_{p,n}) = S^2(-n/(-2n + 1), (-p + 1)/2p, (p + 3)/(2p + 8)) = S^2(n/(2n - 1), (-p + 1)/2p, (p + 3)/(2p + 8))\). If \(n = 0, 1\), then \(K_{m,p,n}(m_{p,n})\) is a lens space, but it is not \(S^2 \times S^1\), because \(m_{p,n} = -2 + n(m - p)^2 = -2 + n(p + 2)^2 \neq 0\). Otherwise, \(0 < n/(2n - 1) < 1\) and \(K_{m,p,n}(m_{p,n}) = S^2(n/(2n - 1), (-p + 1)/2p, (p + 3)/(2p + 8)) = S^2(-1, n/(2n - 1), (p + 1)/2p, (p + 3)/(2p + 8))\).

Since \((p + 1)/2p + (p + 3)/(2p + 8) = 1/2 + 1/2p + 1/2 - 1/(2p + 8) = 1 + 1/2p - 1/(2p + 8) > 1\), \(K_{m,p,n}(m_{p,n})\) is an L-space by Lemma 2.3(2).

Finally assume that \(m \leq -3\). Then \(K_{m,p,n}(m_{p,n}) = S^2(-n/(mn + 1), (-p + 1)/2p, (p - 2m - 1)/(2p - 4m)) = S^2(-1, -n/(mn + 1), (p + 1)/2p, (p - 2m - 1)/(2p - 4m))\).

If \(n = 0\), then \(K_{m,p,n}(m_{p,n})\) is a lens space, but it is not \(S^2 \times S^1\), because \(m_{p,n} = -2 + \frac{n(m - p)^2}{n - 1} \neq 0\). Otherwise, \(0 < n/(mn + 1) < 1\) and \(K_{m,p,n}(m_{p,n}) = S^2(-1, -n/(mn + 1), (p + 1)/2p, (p - 2m - 1)/(2p - 4m))\).
\[m + n(m - p)^2 = m \leq -3.\] Assume \(n \neq 0.\) Then by the assumption \(p \geq 3, m \leq -3\) we have \(0 < -n/(mn + 1) < 1, 0 < (p + 1)/2p < 1\) and \(0 < (p - 2m - 1)/(2p - 4m) = 1/2 - 1/(2p - 4m) < 1.\) Since \((p + 1)/2p + (p - 2m - 1)/(2p - 4m) = 1/2 + 1/2p - 1/2 - 1/(2p - 4m) = 1 + 1/2p - 1/(2p - 4m) > 1,\) Lemma 2.3(2) shows that \(K_{m,n}(m,p,n)\) is an L-space.

(2) Since \(m \leq 0\) and \(p \geq 3, p \neq 2m \pm 1,\) hence \(O \cup c_{m,p}\) is a hyperbolic link; see [13, Theorem 6.21]. Then \(K_{m,p,n}\) is nontrivial for any \(n \neq 0\) [30, 35].

(3) By (1) \(K_{m,p,n}\) is an L-space knot. Since \(O \cup c_{m,p}\) is a hyperbolic link, the hyperbolicity of \(K_{m,p,n}\) for \(|n| > 1\) follows from [2, 21, 37]. Thus \(K_{m,p,n} (|n| > 1)\) is a hyperbolic L-space knot. Let us choose \(c_{m,p}\) and then apply \(n\)-twist along \(c_{m,p}\) to obtain a knot \(K_{m,p,n}\). It remains to show that \(K_{m,p,n}\) and \(K_{m,p,n'}\) are distinct knots. Suppose that \(K_{m,p,n}\) and \(K_{m,p,n'}\) are isotopic for some integers \(n\) and \(n'\) with \(|n|, |n'| > 1.\) Then \((m + n(p - m)^2)\)–, and \((m + n'(p - m)^2)\)–surgeries on \(K_{m,p,n} = K_{m,p,n'}\) produce small Seifert fiber spaces, where \(p - m \geq 3.\) (Note that since \(|n| > 1, mn + 1\) cannot be zero.) Since \(K_{m,p,n}\) is a hyperbolic knot, Lackenby and Meyerhoff [32, Theorem 1.2] prove that the distance \(|m + n(p - m)^2 - (m + n'(p - m)^2)|\) between above two non-hyperbolic surgeries is at most 8. Hence \(|(n - n')(p - m)^2| \leq 8,\) which implies \(n = n'\) because \(p - m \geq 3.\) \(\square\)(Proposition 7.1)

Next we investigate link types of \(O \cup c_{m,p}.\)

**Proposition 7.2** Let \(c_{m,p}\) and \(c_{m',p'}\) be seiferters for \((O, m)\) and \((O, m')\), respectively. Suppose that \(m, m' \leq 0, p, p' \geq 3.\)

1. If \(p - m \neq p' - m',\) then \(O \cup c_{m,p}\) and \(O \cup c_{m',p'}\) are not isotopic. In particular, if \(p \neq p',\) then \(O \cup c_{m,p}\) and \(O \cup c_{m,p'}\) are not isotopic.

2. If \(p - m = p' - m',\) then \(O \cup c_{m,p}\) and \(O \cup c_{m',p'}\) are not isotopic provided that \(|m - m'| > 3.\)

**Proof of Proposition 7.2.** (1) Note that the linking number between \(c_{m,p}\) and \(O\) is \(p - m.\) Hence if \(O \cup c_{m,p}\) is isotopic to \(O \cup c_{m',p'}\) as ordered links, then we have \(p - m = p' - m'.\)

(2) Since \(p \neq 2m \pm 1\) and \(p' \neq 2m' \pm 1,\) both \(O \cup c_{m,p}\) and \(O \cup c_{m',p'}\) are hyperbolic links [13]. Recall that \(c_{m,p}\) is a seiferter for \((O, m)\) and \(c_{m',p'}\) is a seiferter for \((O, m').\) Suppose that \(O \cup c_{m,p}\) and \(O \cup c_{m',p'}\) are isotopic. Then \(c_{m,p}\) is a seiferter for \((O, m')\) as well. Let \(V\) be the solid torus \(S^3 - \text{int}(S^3)\), which contains \(O\) in its interior.
Note that $m$–surgery of $V$ along $O$ yields a Seifert fiber space over the disk with two exceptional fibers of indices $2p, 2p - 4m$, and $m'$–surgery of $V$ along $O$ yields a Seifert fiber space over the disk with two exceptional fibers of indices $2p', 2p' - 4m'$. Since these Seifert fiber spaces contain essential annuli, Gordon and Wu [22, Corollary 1.2] show that $|m - m'| \leq 3$. □ (Proposition 7.2)

Theorem 1.10 follows from Propositions 7.1 and 7.2. □ (Theorem 1.10)

**Remark 7.3** For each seifertor $c_{m,p}$ ($m \leq 0$, $p \geq 3$), $M_{c_{m,p}}(O, m)$ is an L-space. In fact, $M_{c_{m,p}}(O, m)$, which is the limit of $K_{m,p,n}(m_{p,n})$ when $|n|$ tends to $\infty$ (Remark 3.2), is $S^2(-1/m, (-p + 1)/2p, (p - 2m - 1)/(2p - 4m)) = S^2(-1, -1/m, (p + 1)/2p, (p - 2m - 1)/(2p - 4m))$. If $m = -1, 0$, then $M_{c_{m,p}}(O, m)$ is an L-space (Claim 3.4). If $m < -1$, since $(p + 1)/2p + (p - 2m - 1)/(2p - 4m) = 1 + 1/2p - 1/(2p - 4m) > 1$, $M_{c_{m,p}}(O, m)$ is an L-space.

On the other hand, for instance, $M_{c_{1,3}}(O, 3)$ is not an L-space. Indeed, $M_{c_{1,3}}(O, 3) = S^2(-1/3, -1/3, 2/3) = S^2(-2, 2/3, 2/3, 2/3)$, and taking $k = 2$, $a = 1$ in Theorem 2.1(3), we have $(1 - 2/3, 1 - 2/3, 1 - 2/3) = (1/3, 1/3, 1/3) < (1/2, 1/2, 1/2)$. Thus $M_{c_{1,3}}(O, 3)$ is not an L-space.

8 Hyperbolic, L-space knots with tunnel number greater than one

The purpose in this section is to exhibit infinitely many hyperbolic L-space knots with tunnel number greater than one (Theorem 1.13). In [16] Eudave-Muñoz, Jasso and Miyazaki and the author gave Seifert fibered surgeries which do not arise from primitive/Seifert-fibered construction [10].

Let us take unknotted circles $c_a$ and $c_b$ in $S^3 - T_{3,2}$ as illustrated by Figure 8.1. Then as shown in [16] $\{c_a, c_b\}$ is a pair of seifertors for $(T_{3,2}, 7)$, i.e. $c_a$ and $c_b$ become fibers simultaneously in some Seifert fibration of $T_{3,2}(7)$.

Note that the pair $\{c_a, c_b\}$ forms the $(4, 2)$–torus link in $S^3$. Hence $(-1)$–twist along $c_a$ converts $c_a \cup c_b$ into the $(-4, 2)$–torus link. Then we can successively apply $1$–twist along $c_b$ to obtain $(4, 2)$–torus link $c_a \cup c_b$. We denote the images of $c_a$, $c_b$ under twistings along these components by the same symbols $c_a$, $c_b$, respectively.
Let $K_{n,0}$ be a knot obtained from $T_{3,2}$ after the sequence of twistings:

$$(c_a, (-1)-\text{twist}) \rightarrow (c_b, 1-\text{twist}) \rightarrow (c_a, n-\text{twist})$$

Then $K_{n,0} = K(2, -n, 1, 0)$ in [16, Proposition 4.11]. See Figure 8.2.

Similarly, let $K_{0,n}$ be a knot obtained from $T_{3,2}$ after the sequence of twistings:

$$(c_a, (-1)-\text{twist}) \rightarrow (c_b, n + 1-\text{twist})$$

Then $K_{0,n} = K(2, 0, 1, -n)$ in [16, Proposition 4.11]. See Figure 8.2.

Figure 8.1: $\{c_a, c_b\}$ is a pair of seiferters for $(T_{3,2}, 7)$.

Figure 8.2: Seifert surgeries $(K_{n,0}, 196n + 71)$ and $(K_{0,n}, 100n + 71)$; each vertex corresponds to a Seifert surgery and each edge corresponds to a single twist along a seiferter.

Theorem 1.13 follows from Theorem 8.1 below.
Theorem 8.1  

(1) \(\{K_{n,0}\}_{n \in \mathbb{Z}}\) is a set of mutually distinct hyperbolic L-space knots with tunnel number two.

(2) \(\{K_{0,n}\}_{n \in \mathbb{Z}\setminus\{-1\}}\) is a set of mutually distinct hyperbolic L-space knots with tunnel number two.

Proof of Theorem 8.1. We begin by recalling the following result which is a combination of Propositions 3.2, 3.7 and 3.11 in [16].

Lemma 8.2  

(1) \(K_{n,0}\) is a hyperbolic knot with tunnel number two, and \(K_{n,0}(196n + 71)\) is a Seifert fiber space \(S^2((11n + 4)/(14n + 5), -2/7, 1/2)\).

(2) \(K_{0,n}\) is a hyperbolic knot with tunnel number two if \(n \neq -1\), and \(K_{0,n}(100n + 71)\) is a Seifert fiber space \(S^2(-(3n + 2)/(10n + 7), 4/5, 1/2)\).

Lemma 8.3  

(1) If \(K_{n,0}\) and \(K_{n',0}\) are isotopic, then \(n = n'\).

(2) If \(K_{0,n}\) and \(K_{0,n'}\) are isotopic, then \(n = n'\).

Proof of Lemma 8.3: (1) Suppose that \(K_{n,0}\) is isotopic to \(K_{n',0}\). Then \(K_{n,0}(196n + 71)\) and \(K_{n,0}(196n' + 71)\) are both Seifert fiber spaces. Since \(K_{n,0}\) is hyperbolic, Theorem 1.2 in [32] implies that \(|196n + 71 - (196n' + 71)| = |196(n - n')| \leq 8\). Hence we have \(n = n'\). (2) follows in a similar fashion. \(\square\) (Lemma 8.3)

Let us prove that \(K_{n,0}\) and \(K_{0,n}\) are L-space knots for any integer \(n\).

Lemma 8.4  

(1) \(K_{n,0}(196n + 71)\) is an L-space for any integer \(n\).

(2) \(K_{0,n}(100n + 71)\) is an L-space for any integer \(n\).

Proof of Lemma 8.4: (1) Note that \(K_{n,0}(196n + 71) = S^2((11n + 4)/(14n + 5), -2/7, 1/2) = S^2(-1, (11n + 4)/(14n + 5), 5/7, 1/2)\). Since \(0 < (11n + 4)/(14n + 5) < 1\) for any \(n \in \mathbb{Z}\) and \(5/7 + 1/2 = 1\), Lemma 2.3(2) shows that \(K_{n,0}(196n + 71)\) is an L-space for any integer \(n\). This proves (1).

(2) As above first we note that \(K_{0,n}(100n + 71) = S^2(-(3n + 2)/(10n + 7), 4/5, 1/2) = S^2(-1, (7n + 5)/(10n + 7), 4/5, 1/2)\). Since \(0 < (7n + 5)/(10n + 7) < 1\) for any \(n \in \mathbb{Z}\) and \(4/5 + 1/2 = 1\), Lemma 2.3(2) shows that \(K_{0,n}(100n + 71)\) is an L-space for any integer \(n\). \(\square\) (Lemma 8.4)

Now Theorem 8.1 follows from Lemmas 8.2, 8.3 and 8.4. \(\square\) (Theorem 8.1)

Question 8.5  Does there exist a hyperbolic L-space knot with tunnel number greater than two? More generally, for a given integer \(p\), does there exist a hyperbolic L-space knot with tunnel number greater than \(p\)?
9 Questions

9.1 Characterization of twistings which yield infinitely many L-space knots.

For knots $K$ with Seifert surgery $(K, m)$, Theorems 1.4, 1.5, 1.6 and Corollary 1.7 characterize seiferters which enjoy the desired property in Question 1.1.

The next proposition, which is essentially shown in [25, 26], describes yet another example of twistings which yield infinitely many L-space knots.

**Proposition 9.1** (L-space twisted satellite knots) Let $k$ be a nontrivial knot with L-space surgery $(k, 2g - 1)$, where $g$ denotes the genus of $k$, and $K$ a satellite knot of $k$ which lies in $V = N(k)$ with winding number $w$. Suppose that $V(K; m)$ is a solid torus for some integer $m \geq w^2(2g - 1)$. Let $c$ be the boundary of a meridian disk of $V$, and $K_n$ a knot obtained from $K$ by $n$--twist along $c$. Then $K_n$ is an L-space knot for any $n \geq 0$. See Figure 9.1.

**Proof of Proposition 9.1.** Recall that $K_n(m + nw^2) = k((m + nw^2)/w^2) = k(m/w^2 + n)$ [20]. Since $k(2g - 1)$ is an L-space and $m/w^2 \geq 2g - 1$, [48, Proposition 9.6] ensures that $k(m/w^2 + n)$ is also an L-space if $n \geq 0$. Hence $K_n$ is an L-space knot provided $n \geq 0$. □(Proposition 9.1)

![Figure 9.1](image)

**Remark 9.2** (1) In Proposition 9.1, the knot $K$ in the solid torus $V$ is required to have a cosmetic surgery: $V(K; m) \cong S^1 \times D^2$. The cosmetic surgery of the solid torus is well-understood by [18, 5].

(2) Twisting operation described in Proposition 9.1 can be applied only for satellite knots and resulting knots after the twistings are also satellite knots.
In Proposition 9.1, the knot $k$ is assumed to be nontrivial. If $k$ is a trivial knot in $S^3$, then $K(m) = (S^3 - \text{int}V) \cup V(K;m)$ is a lens space, hence $(K, m)$ is an L-space surgery. It is easy to see that $c$ is a seiferter for $(K, m)$.

For further possibility, weaken a condition of seiferter to obtain a notion of “pseudo-seiferter” as follows.

**Definition 9.3 (pseudo-seiferter)** Let $(K, m)$ be a Seifert surgery. A knot $c$ in $S^3 - N(K)$ is called a pseudo-seiferter for $(K, m)$ if $c$ satisfies (1) and (2) below.

1. $c$ is a trivial knot in $S^3$.
2. $c$ becomes a “cable” of a fiber in a Seifert fibration of $K(m)$ and the preferred longitude $\lambda$ of $c$ in $S^3$ becomes the cabling slope of $c$ in $K(m)$.

We do not know if a pseudo-seiferter exists, but if $(K, m)$ admits a pseudo-seiferter, it behaves like a seiferter in the following sense. Let $V$ be a fibered tubular neighborhood of a fiber $t$ and $c$ is a cable in $V$. Then the result of a surgery (corresponding to $n$--twist) on $c$ of $V$ is again a solid torus, and this surgery is reduced to a surgery on the fiber $t$ which is a core of $V$. Hence $K_n(m_n)$ is a (possibly degenerate) Seifert fiber space. This suggests that a pseudo-seiferter is also a candidate for an unknotted circle described in Question 1.1.

We would like to ask the following question for non-satellite knots.

**Question 9.4** Let $K$ be a non-satellite knot and $K_n$ a knot obtained from $K$ by $n$--twist along an unknotted circle $c$ in $S^3 - K$. Suppose that the twist family $\{K_n\}$ contains infinitely many L-space knots.

1. Does $K$ admit a Seifert surgery $(K, m)$ for which $c$ is a seiferter?
2. Does $K$ admit a Seifert surgery $(K, m)$ for which $c$ is a seiferter or a pseudo-seiferter?

### 9.2 L-space knots and strong invertibility.

A knot is said to be **strongly invertible** if there exists an orientation preserving involution of $S^3$ which fixes the knot setwise and reverses orientation. Known L-space knots are strongly invertible, so it is natural to ask:
Problem 9.5 (Watson) Are L-space knots strongly invertible?

In [13] an “asymmetric seiferter” defined below is essentially used to find Seifert fibered sugary on knots with no symmetry.

Definition 9.6 (asymmetric seiferter) A seiferter $c$ for a Seifert surgery $(K, m)$ is said to be symmetric if we have an orientation preserving diffeomorphism $f : S^3 \rightarrow S^3$ of finite order with $f(K) = K, f(c) = c$; otherwise, $c$ is called an asymmetric seiferter.

Combining [13, Theorem 7.3] and Theorem 1.4, we obtain:

Proposition 9.7 Let $(K, m)$ be a Seifert fibered surgery on a non-satellite knot with an asymmetric seiferter $c$ which becomes an exceptional fiber. Suppose that $M_c(K, m)$ is an L-space. Then there is a constant $N$ such that $K_n$, a knot obtained from $K$ by $n$–twist along $c$, is a hyperbolic L-space knot with no symmetry for any $n \leq N$ or $n \geq N$.

If $c$ is a seiferter for $(T_{p,q}, pq)$ which becomes a degenerate fiber in $T_{p,q}(pq)$, then $c$ is a meridian of $T_{p,q}$ or $T_{p,q} \cup c$ is a hyperbolic link in $S^3$; see [13, Theorem 3.19(3)]. Hence the argument in the proof of Theorem 7.3 in [13] and Theorem 1.6(2) enable us to show:

Proposition 9.8 If $c$ is an asymmetric seiferter for $(T_{p,q}, pq)$ which becomes a degenerate fiber in $T_{p,q}(pq)$, then there is a constant $N$ such that $K_n$ is a hyperbolic L-space knot with no symmetry for any $n \leq N$ or $n \geq N$.

For the asymmetric seiferter $c = c'_1$ for $(K, m) = (P(-3,3,5), 1)$ given in [13, Lemma 7.5], $M_c(K, m)$ is not an L-space and $c$ does not satisfy the hypothesis of Proposition 9.7.

Question 9.9 Does there exist an asymmetric seiferter described in Propositions 9.7 and 9.8?

References


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Department of Mathematics, Nihon University, 3-25-40 Sakurajosui, Setagaya-ku, Tokyo 156-8550, Japan
aderuelle@math.chs.nihon-u.ac.jp
http://www.math.chs.nihon-u.ac.jp/~motegi/