TIGHT FIBERED KNOTS AND BAND SUMS

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Abstract. We give a short proof that if a non-trivial band sum of two knots results in a tight fibered knot, then the band sum is a connected sum. In particular, this means that any prime knot obtained by a non-trivial band sum is not tight fibered. Since a positive L-space knot is tight fibered, a non-trivial band sum never yields an L-space knot. Consequently, any knot obtained by a non-trivial band sum cannot admit a finite surgery.

For context, we exhibit two examples of non-trivial band sums of tight fibered knots producing prime knots: one is fibered but not tight, and the other is strongly quasipositive but not fibered.

1. Introduction

Let $K_1 \sqcup K_2$ be a 2-component split link in the 3-sphere S^3. Let $\beta\colon [0,1] \times [0,1] \to S^3$ be an embedding such that $\beta([0,1]\times[0,1])\cap K_1 = \beta([0,1]\times\{0\})$ and $\beta([0,1]\times[0,1])\cap K_2 = \beta([0,1]\times\{1\})$. Then we obtain a knot $K_1 \natural_\beta K_2$ by replacing $\beta([0,1] \times \{0,1\})$ in $K_1 \cup K_2$ with $\beta(\{0,1\} \times [0,1])$. We call $K_1 \natural_\beta K_2$ a *band sum* of K_1 and K_2 with the band β. In the following, for simplicity, we use the same symbol β to denote the image $\beta([0,1] \times [0,1])$. We say that a band sum is *trivial* if one of K_1 and K_2, say K_2, is the unknot and the band β gives just a connected sum. If the band sum is trivial, then obviously $K_1 \natural_\beta K_2 = K_1$. The converse also holds, i.e. if $K_1 \natural_\beta K_2 = K_1$, then the band sum is trivial; see [3, 21]. A band sum is regarded as a natural generalization of the connected sum, and many prime knots are obtained by band sums; see [10].

We say a fibered knot in S^3 is *tight* if, as an open book for S^3, it supports the positive tight contact structure on S^3. A knot in S^3 is *strongly quasipositive* if it is the boundary of a *quasipositive* Seifert surface, a special kind of Seifert surface obtained from parallel disks by attaching positive bands in a particular kind of braided manner, for a more precise definition see e.g. [20, 61.Definition]. It is shown by Rudolph [19, Characterization Theorem] (cf. [20, 90.Theorem]) that a knot K is strongly quasipositive if and only if it has a Seifert surface that is a subsurface of the fiber of some positive torus knot. Hedden proved that tight fibered knots are precisely the fibered strongly quasipositive knots [6, Proposition 2.1]; Baader-Ishikawa [1, Theorem 3.1] provides an alternative proof. From this correspondence, one may observe that a connected sum of two tight fibered knots is again a tight fibered knot. (Of course this follows more directly from the definitions of connected sums of contact manifolds and contact structures supported by open books, e.g. [2, 4].)

The aim of this note is to prove:

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Theorem 1.1. *If a tight fibered knot in S^3 is a non-trivial band sum, then the band sum expresses the knot as a connected sum. In particular, it is not prime.*

Corollary 1.2. *If a prime knot in S^3 is a non-trivial band sum, then it is not a tight fibered knot.*

A knot K in the 3–sphere S^3 is called an *L-space knot* if it admits a nontrivial Dehn surgery yielding an L-space, a rational homology sphere whose Heegaard Floer homology is as simple as possible [18]. It is a *positive* (resp. *negative*) L-space knot if a positive (resp. negative) Dehn surgery yields an L-space. In [11] Krcatovich proves that if a knot is a connected sum of non-trivial knots, then it is not an L-space knot. Since positive L-space knots are special types of tight fibered knots by [6] with [16, 5] (cf. [7]), Theorem 1.1 allows us to generalize Krcatovich's result to non-trivial band sums:

Corollary 1.3. *If a knot in S^3 is a non-trivial band sum, then it is not an L-space knot.*

Since lens spaces, and more generally 3–manifolds with finite fundamental group, are L-spaces [18], Corollary 1.3 immediately implies:

Corollary 1.4. *Any knot obtained by a non-trivial band sum does not admit a finite surgery.*

Recall from [6, 1] that the set of strongly quasipositive, fibered knots coincides with that of tight fibered knots. Thus Theorem 1.1 says that any prime knot obtained by a non-trivial band sum fails to satisfy at least one of conditions (i) K is strongly quasipositive, and (ii) K is fibered.

In fact, we demonstrate:

Proposition 1.5.
(1) *There exist tight fibered knots K_1 and K_2, together with a band β, such that $K_1 \natural_\beta K_2$ is prime, strongly quasipositive, but not fibered.*
(2) *There exist tight fibered knots K_1 and K_2, together with a band β, such that $K_1 \natural_\beta K_2$ is prime, fibered, but not tight* (*hence not strongly quasipositive*).

The connected sum of two tight fibered knots is tight fibered. On the other hand we show:

Proposition 1.6. *For any tight fibered knots K_1 and K_2, there exists a band β such that $K_1 \natural_\beta K_2$ is fibered, but not tight* (*hence not strongly quasipositive*).

2. Proofs of Theorem 1.1 and Corollary 1.3

Proof of Theorem 1.1. Assume a knot $K = K_1 \natural_\beta K_2$ in S^3 is a non-trivial band sum of the split link $K_1 \sqcup K_2$ of knots K_1 and K_2 with a band β. Then $g(K) \geq g(K_1) + g(K_2)$ since genus is superadditive for band sums by Gabai [3, Theorem 1] and Scharlemann [21, 8.4 Theorem]. Since K is concordant to $K_1 \# K_2$ by Miyazaki [14, Theorem 1.1], $\tau(K) = \tau(K_1 \# K_2)$ where τ is the Ozváth-Szábo concordance invariant [17]. Furthermore, additivity of τ under connected sum [17, Proposition 3.2] shows $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$. If we also suppose that $\tau(K) = g(K)$, then because τ gives a lower bound on genus for all knots in S^3 [17, Corollary 1.3], we will have the string of inequalities:

$$\tau(K) = g(K) \geq g(K_1) + g(K_2) \geq \tau(K_1) + \tau(K_2) = \tau(K).$$

It then follows that $g(K) = g(K_1) + g(K_2)$.

Now for the proposition, assume the band sum K is a tight fibered knot. It follows from Hedden [6, Proposition 2.1] and also Baader-Ishikawa [1, Theorem 3.1] that K is strongly quasipositive. Hence Livingston [13, Theorem 4] shows that $\tau(K) = g(K)$. It then follows from the above calculation that we have $g(K) = g(K_1) + g(K_2)$.

Now by Kobayashi [9, Theorem 2], both K_1 and K_2 are fibered, and the banding expresses K as the connected sum of K_1 and K_2, i.e. there is a 2–sphere which split K_1 and K_2 and intersects β in a single arc. The proof of [9, Proposition 4.1] clarifies this last point. In particular, K cannot be prime. For otherwise, K_1 or K_2 would be a trivial knot and the band sum is trivial, a contradiction to the assumption. □

Proof of Corollary 1.3. Let K be an L-space knot. By mirroring, we may assume it is a positive L-space knot. Then by combining Hedden [6] with Ni & Ghiggini [16, 5] (cf. [7]), K is a tight fibered knot. Krcatovich [12, 11] shows that K must be prime. Now Theorem 1.1 implies K cannot be a non-trivial band sum. Note that if K is a positive L-space knot, then the equality $\tau(K) = g(K)$ follows by [18]. □

3. A family of prime tangles

In Lemma 4.3 we show Example 4.2 of a band sum is prime by demonstrating that it has a prime tangle decomposition and then employing the work of Nakanishi [15]. The two tangles involved both have a similar form which we generalize to the family of $(n+1)$–strand tangles $T^n(\tau)$ based off a two-strand tangle τ. (See Figure 3.1 and the discussion below.) In Proposition 3.3 we provide hypotheses that ensure the tangle $T^n(\tau)$ is prime for $n \geq 1$.

For our purposes, an *n–string tangle* is a proper embedding of a disjoint union of n arcs into a ball considered up to proper isotopy; n is a positive integer. Recall that a tangle T is *prime* if it satisfies the following conditions:

(1) T is not a trivial 1–string tangle.
(2) T is *locally trivial*: Any 2–sphere that transversally intersects T in just two points bounds a ball in which T is a trivial 1–string tangle.
(3) T is *non-splittable*: Any properly embedded disk does not split T.
(4) T is *indivisible*: Any properly embedded disk that transversally intersects T in a single point divides the tangle into two subtangles, at least one of which is the trivial 1–string tangle. In the following we say that such a disk is also *boundary-parallel*.

Given a two-strand tangle τ with fixed boundary so that the endpoints on the left belong to different arcs, define the $(n+1)$–strand tangle $T^n(\tau)$ as in Figure 3.1 for non-negative integers n. For convenience, let us take $k = T^0(\tau) = (B, k)$, $T = T^1(\tau) = (B, k \cup a)$, and $T' = T^2(\tau) = (B, k \cup a \cup a')$ as also shown in Figure 3.1. Here B is the 3–ball.

Lemma 3.1. *If $T = T^1(\tau) = (B, k \cup a)$ is locally non-trivial, then k is a knotted arc and (B, τ) is locally non-trivial, containing a summand of the knotted arc k.*

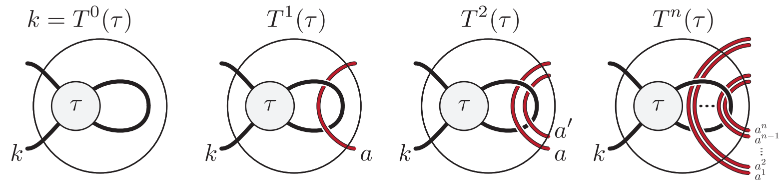

FIGURE 3.1. The tangles $k = T^0(\tau)$, $T = T^1(\tau) = (B, k \cup a)$, $T' = T^2(\tau) = (B, k \cup a \cup a')$, and $T^n(\tau) = (B, k \cup a^1 \cup a^2 \cup \dots a^{n-1} \cup a^n)$.

Remark 3.2. This summand is not necessarily proper. For example, the tangle τ could be homeomorphic to a split tangle consisting of k and another arc.

Proof. Assume the tangle $(B, k \cup a)$ is locally non-trivial. Since a is a trivial arc, any local knotting would have to occur in k. Let S be a sphere bounding a ball B_S that intersects $k \cup a$ in a knotted subarc of k. By replacing S with a smaller one if necessary, we may assume the closure of the knotted 1–string tangle $(B_S, B_S \cap k)$ is a prime knot. Observe that the arc a together with an arc in ∂B bounds a disk δ that intersects k exactly once. We will inductively isotop S and k leaving a invariant so that $S \cap \delta = \emptyset$.

Assume that $S \cap \delta \neq \emptyset$. Because S is disjoint from a, $S \cap \delta$ is a collection of circles. A circle of $S \cap \delta$ that is innermost in δ bounds a subdisk $\delta' \subset \delta$.

If there is such a disk δ' that is disjoint from k, then the circle $\partial\delta'$ also bounds a subdisk $\sigma \subset S$ that is disjoint from k. Since $B - \mathcal{N}(k)$ is irreducible (because it is the exterior of the knot K, the closure of k) and $\delta' \cup \sigma \subset B - \mathcal{N}(k)$, we may isotop σ to δ' (in the complement of $k \cup a$) and then further isotop S to reduce $|S \cap \delta|$. Perform such isotopies until every circle of $S \cap \delta$ bounds a subdisk of δ that intersects k; in particular, the circles $S \cap \delta$ are concentric circles in δ about the single intersection point $\delta \cap k$.

Now if $S \cap \delta \neq \emptyset$, then the innermost one in δ bounds a disk $\delta' \subset \delta$ that meets k in a single point and is contained in B_S. Then $\partial\delta'$ divides S into two disks σ and σ' that each intersect k once and δ' divides B_S into two balls bounded by spheres $\delta' \cup \sigma$ and $\delta' \cup \sigma'$. Since we have chosen S so that $(B_S, B_S \cap k)$ is a prime knot, one of these balls intersects k in a trivial arc. We may use these balls to isotop B_S along with $k \cap B_S$ into a collar neighborhood of δ' and then further to be disjoint from δ' (and δ).

Since $S \cap \delta = \emptyset$, B_S is contained in $(B - \mathcal{N}(\delta), k \cup a) \cong (B, \tau)$. Hence τ contains a locally knotted arc which is the summand of k sectioned off by B_S. $\square$

Proposition 3.3. *If k is a non-trivial arc without any proper summand in B, and τ is locally trivial, then $T^n(\tau)$ is prime for $n \geq 1$.*

Proof. We proceed to check the four conditions of primeness for $T^n(\tau)$:

(1) This is obvious since $T^n(\tau)$ is an $n+1$–string tangle and $n \geq 1$.

(2) If $T^n(\tau)$ were locally non-trivial, then since the arcs $a^1, \dots, a^n$ are trivial arcs, the local knotting would have to occur in the component k. Thus the 2–string tangle $T = T^1(\tau) = (B, k \cup a)$ would also be locally non-trivial (where we let $a = a^1$). Then Lemma 3.1 shows that this implies

that τ would have to be a locally non-trivial 2–string tangle containing a summand of k. This implies that τ is locally non-trivial, contradicting the assumption.

(3) Assume $T^n(\tau)$ is splittable. Then there is a disk D that separates one component of $T^n(\tau) = (B, k \cup a^1 \cup \cdots \cup a^n)$ from another. Hence D must separate k from the arc a^i for some i. Since the arcs $a^1, \ldots, a^n$ are mutually isotopic in the complement of k, we may assume $k \cup a^1$ is splittable. However, this implies that $T^1(\tau) = (B, k \cup a^1)$ is locally non-trivial since k is a non-trivial arc, a contradiction to (2).

(4) Assume $T^n(\tau)$ is divisible. Then there is a disk D transversally intersecting $T^n(\tau)$ in just one point so that D is not boundary-parallel. Since the arcs $a^1, \ldots, a^n$ are mutually isotopic, we may assume either D intersects k or D intersects a^1.

If D intersects k, then it must cut off a trivial arc from k since k has no proper summand. Because D is not boundary-parallel, some arc of $a^1, \ldots, a^n$ must be on this side, say a^i. But then $T^1(\tau) = (B, k \cup a^i)$ is locally non-trivial, a contradiction to (2).

If D intersects a^1, then the arcs $a^2, \ldots, a^n$ and k must all be on the same side of D. Otherwise, or some $i \geq 2$, $T^1(\tau) = (B, k \cup a^i)$ would be a splittable 2–string tangle contrary to (3). But now since a^1 is a trivial arc, D must be boundary-parallel to the side that does not contain $a^2, \ldots, a^n$ and k. Again this is a contradiction. □

4. Examples

In this section, we give examples that prove Propositions 1.5 and 1.6. Examples 4.2 and 4.4 give Proposition 1.5(1) and (2) respectively. Proposition 1.6 follows from Example 4.5.

4.1. **A prime, non-fibered, strongly quasipositive banding of tight fibered knots.** We show that there exists a banding $K = K_1 \natural_\beta K_2$ of tight fibered knots K_1 and K_2 which is prime and strongly quasipositive; furthermore, K has the Alexander polynomial of an L-space knot. It follows from Theorem 1.1 (or by Kobayashi [9]) that K cannot be fibered, and thus it is not an L-space knot [16, 5] (cf. [7]). This is punctuates Corollary 1.3.

Lemma 4.1. *If $K = K_1 \natural_\beta K_2$ and $g(K) = g(K_1) + g(K_2)$ then $\Delta_K(t) = \Delta_{K_1}(t)\Delta_{K_2}(t)$.*

Proof. By Gabai [3] and Scharlemann [21], when $g(K) = g(K_1) + g(K_2)$ for the band sum $K = K_1 \natural_\beta K_2$, then there are minimal genus Seifert surfaces F, F_1, F_2 for the knots K, K_1, K_2 such that F is the union of F_1, F_2 and the band β. Since F_1 and F_2 are separated by a sphere, F has a Seifert matrix that is block diagonal of the Seifert matrices for F_1 and F_2. The result then follows. □

Proposition 1.5(1) follows from the example below.

Example 4.2. Let K_1 be the $(2,3)$–torus knot $T_{2,3}$ and K_2 the $(2,1)$–cable of the $(2,3)$–torus knot $T_{2,3}^{2,1}$; K_2 wraps twice in a longitudinal direction and once in a meridional direction along the companion $T_{2,3}$. Note that K_1 and K_2 are both strongly quasipositive fibered knots, and hence tight fibered knots [6]. The left-hand side of Figure 4.1 shows the split link $K_1 \sqcup K_2$. The right hand side of Figure 4.1 shows the banding $K = K_1 \natural_\beta K_2$ which, by virtue of its presentation is also strongly quasipositive.

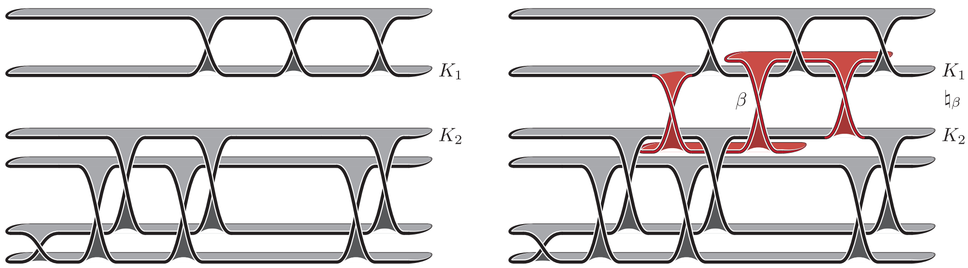

FIGURE 4.1. The $(2,3)$–torus knot K_1 and the $(2,1)$–cable of the $(2,3)$–torus knot K_2 are strongly quasipositive knots. They may be banded together to form another strongly quasipositive knot $K = K_1 \natural_\beta K_2$.

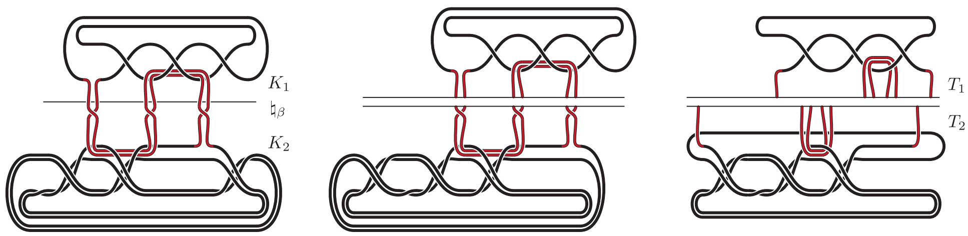

FIGURE 4.2. The splitting sphere of $K_1 \sqcup K_2$ divides $K = K_1 \natural_\beta K_2$ into two three strand tangles, T_1 and T_2. The right-hand picture of the tangles gives isotopic versions in the center.

By Lemma 4.1, it now follows that $\Delta_K(t) = \Delta_{K_1}(t)\Delta_{K_2}(t) = (t-1+t^{-1})(t^2-1+t^{-2})$. Indeed, let us note that we constructed K so that $\Delta_K(t)$ is also the Alexander polynomial of an L-space knot, namely the L-space knots $T_{3,4}$ and $T_{2,3}^{2,3}$, i.e. the $(3,4)$–torus knot and the $(2,3)$–cable of $(2,3)$ torus knot.

Lemma 4.3 below uses tangle theory to shows that $K = K_1 \natural_\beta K_2$ is a prime knot. Hence by Theorem 1.1 (or by Kobayashi [9]), K is not fibered.

Lemma 4.3. *The knot $K = K_1 \natural_\beta K_2$ on the right-hand side of Figure 4.1 is a prime knot.*

Proof. Discarding the Seifert surface followed by a small isotopy presents $K = K_1 \natural_\beta K_2$ as on the left-hand side of Figure 4.2. The sphere separating K_1 and K_2 that intersects K in 6 points (shown first as a horizontal line) splits K into two 3–string tangles T_1 and T_2 as shown in Figure 4.2. Then Figure 4.3 shows, for each $i = 1, 2$, the results of further isotopies expressing T_i as the tangle $T^2(\tau_i) = k_i \cup a_i \cup a_i'$ for some 2–string tangle τ_i such that k_i is a knotted arc of knot type K_i while a and a' are trivial arcs that are isotopic in the complement of k_i. (See Figure 3.1.) Note that the knot K_i is the closure of τ_i while the arc $k(\tau_i)$ is the "half-closure" of τ_i.

Since K_i is a non-trivial prime knot for each $i = 1, 2$, then the arcs k_i are non-trivial without any proper summand. Because τ_1 is a trivial tangle, it is locally trivial. Because k_2 has no proper summand, if τ_2 were locally non-trivial, the local knotting would have the knot type of k_2. However since the two one-strand subtangles of τ_2 are a trivial arc and a knotted arc which one may identify as having the knot type of 8_{20} which is not the type of k_2, τ_2 is locally trivial. (Figure 4.3 highlights

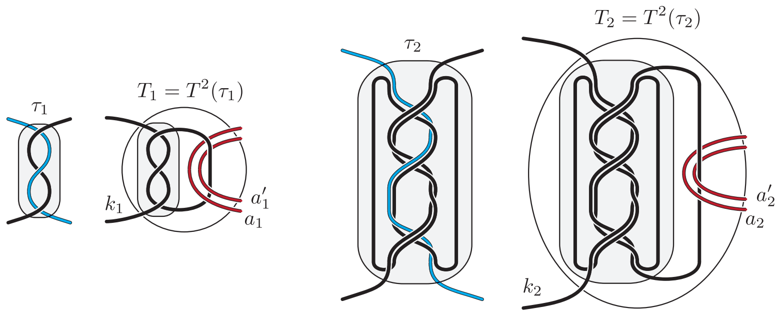

FIGURE 4.3. A further isotopy of the two tangles T_1 and T_2 exhibits them as the tangles $T^2(\tau_1)$ and $T^2(\tau_2)$.

one strand in each τ_1 and τ_2 so that the other strand is more easily discerned.) Thus Proposition 3.3 shows that T_1 and T_2 are both prime tangles. Work of Nakanishi then implies that K is a prime knot [15]. □

4.2. **A prime, fibered, non-strongly quasipositive banding of tight fibered knots.** We show that there exists a banding $K = K_1 \natural_\beta K_2$ of tight fibered knots K_1 and K_2 which is prime, fibered, but not tight, and hence not strongly quasipositive.

Example 4.4. Take $K_1 = T_{2,3}$ and the trivial knot K_2, each of which is a tight fibered knot. Let $K_1 \sqcup K_2$ be a split link, and β a band given in Figure 4.4. Then the band sum $K = K_1 \natural_\beta K_2$ is the prime knot 8_{10}. (This band sum presentation of 8_{10} was given by Kobayashi [10].) Recall that K is a Montesinos knot $M(\frac{1}{2}, \frac{1}{3}, \frac{2}{3})$ and it is easy to see that K is a plumbing of two torus knots $T_{2,5}$ and $T_{2,-3}$. Hence it is a fibered knot [22, Theorem 1]. By Theorem 1.1 K is not tight, and hence it is not strongly quasipositive.

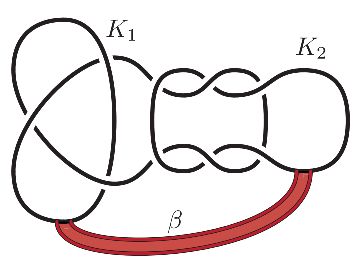

FIGURE 4.4. $K_1 = T_{3,2}$, $K_2 =$ unknot, and $K = K_1 \natural_\beta K_2$ is 8_{10}.

4.3. **A non-prime, fibered, non-strongly quasipositive banding of tight fibered knots.** Recall that the connected sum of tight fibered knots is always tight fibered. In this subsection we show that for any tight fibered knots K_1 and K_2, we can take a band β so that $K_1 \natural_\beta K_2$ is fibered, but not tight, and hence not strongly quasipositive.

Example 4.5. Figure 4.5 shows a band β for a split link of any two knots K_1 and K_2 so that the band sum $K = K_1 \natural_\beta K_2$ produces the connected sum $K = K_1 \# 3_1 \# \overline{3_1} \# K_2$. Since the square knot $3_1 \# \overline{3_1}$ is fibered, if K_1 and K_2 are fibered then K will be fibered. However since $3_1 \# \overline{3_1}$ is a non-trivial ribbon knot, $\tau(3_1 \# \overline{3_1}) = 0$, hence $\tau(K_1 \# 3_1 \# \overline{3_1} \# K_2) = \tau(K_1 \# K_2) \leq g(K_1 \# K_2) < g(K_1 \# 3_1 \# \overline{3_1} \# K_2)$. This shows K cannot be strongly quasipositive, and in particular it is not tight. Thus, choosing K_1 and K_2 to be tight fibered knots gives the example.

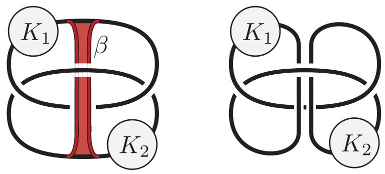

FIGURE 4.5. The band β for the split link $K_1 \sqcup K_2$ produces the band sum $K_1 \natural_\beta K_2 = K_1 \# 3_1 \# \overline{3_1} \# K_2$.

5. Further discussion and questions

Our results and proofs lead to a few natural questions.

5.1. **Band sums of strongly quasipositive knots.**

Question 5.1. *If $K = K_1 \natural_\beta K_2$ is a strongly quasipositive knot, must K_1 and K_2 be strongly quasipositive?*

Since $\tau(K) = g(K)$ for strongly quasipositive knots, the argument in the proof of Theorem 1.1 enables us to conclude that $g(K) = g(K_1) + g(K_2)$. Then the equality implies that there are minimal genus Seifert surfaces F_1, F_2 for the knots K_1, K_2 that are disjoint from the band β so that $F = F_1 \cup \beta \cup F_2$ is a minimal genus Seifert surface for K; see Gabai [3, Theorem 1] and Scharlemann [21, 8.5 Remark]. If F is a quasipositive Seifert surface, then it follows that F_1 and F_2 are also quasipositive Seifert surfaces (because they are subsurfaces of F) and hence K_1 and K_2 are strongly quasipositive knots.

From this point of view, an affirmative answer to this question would follow from an affirmative answer to the following.

Question 5.2. *If F is a minimal genus Seifert surface for a strongly quasipositive knot, must F be a quasipositive surface?*

Let us note that the Kakimizu complex for minimal genus Seifert surfaces for a knot is connected [8]. Thus if the answer to this question is negative, then there is a strongly quasipositive knot K with a quasipositive Seifert surface Q and a non-quasipositive minimal genus Seifert surface F such that $F \cap Q = \partial F = \partial Q = K$.

5.2. **Band sums of split links.** The band sum operation can be generalized as follows. Start with a split link with n components $K_1, \ldots, K_n$ (where for each component there is a sphere separating it from all of the other components), and connect the components via $n-1$ pairwise disjoint bands to obtain a knot K. We call K a *band sum* of $K_1, \ldots, K_n$; see [14]. As an analogy of the case $n = 2$, we say that a band sum is *trivial* if K coincides with one of K_i ($1 \le i \le n$).

Question 5.3.
(1) *If a prime knot K in S^3 is a non-trivial band sum of $K_1, \ldots, K_n$, then can K be a tight fibered knot?*
(2) *If a knot K in S^3 is a non-trivial band sum of $K_1, \ldots, K_n$, then can K be an L-space knot?*

We expect a negative answer to both of these questions. Since the relation $g(K_1)+\cdots+g(K_n) \ge \tau(K_1)+\cdots+\tau(K_n) = \tau(K)$ holds, if one follows the scheme of our proof of Theorem 1.1, it becomes a question of whether the "superadditivity of genus" for such band sums holds true and whether there is a result similar to Kobayashi's work on fibering and band sums.

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