TWIST FAMILIES OF L-SPACE KNOTS, THEIR GENERA, AND SEIFERT SURGERIES

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Abstract. Conjecturally, there are only finitely many Heegaard Floer L-space knots in S^3 of a given genus. We examine this conjecture for twist families of knots $\{K_n\}$ obtained by twisting a knot K in S^3 along an unknot c in terms of the linking number ω between K and c. We establish the conjecture in case of $|\omega| \neq 1$, prove that $\{K_n\}$ contains at most three L-space knots if $\omega = 0$, and address the case where $|\omega| = 1$ under an additional hypothesis about Seifert surgeries. To that end, we characterize a twisting circle c for which $\{(K_n, r_n)\}$ contains at least ten Seifert surgeries. We also pose a few questions about the nature of twist families of L-space knots, their expressions as closures of positive (or negative) braids, and their wrapping about the twisting circle.

1. Introduction

Let M be a rational homology 3–sphere. Then its Heegaard Floer homology $\widehat{\mathrm{HF}}(M)$ satisfies $\widehat{\mathrm{HF}}(M) \geq |H_1(M;\mathbb{Z})|$. We call M an *L-space* if this is actually an equality, $\widehat{\mathrm{HF}}(M) = |H_1(M;\mathbb{Z})|$. As its name suggests, lens spaces other than $S^1 \times S^2$ are L-spaces. More generally, the set of L-spaces include all 3–manifolds with finite fundamental group [55, Proposition 2.3] and many Seifert fibered spaces [55, 40]. A knot K in the 3–sphere S^3 is called an *L-space knot* if $K(r)$, the result of r–surgery on K, is an L-space for some $r \in \mathbb{Q}$. Specifically we call a non-trivial L-space knot *positive* or *negative* according to the sign of r; only the unknot has both positive and negative L-space surgeries. Note that if K is a negative L-space knot, then its mirror image is a positive L-space knot. Frequently it is preferable to restrict the definition of L-space knots to mean just positive L-space knots, however for our purposes here it is convenient to make the broader definition. Since the trivial knot, torus knots, and Berge knots in general [4] admit surgeries yielding lens spaces, these knots are fundamental examples of L-space knots; see also [44].

Our motivation for this article takes root in the following conjecture due to Hedden-Watson [30, Conjecture 6.7]. Recall that Ozsváth-Szabó [53] and Rasmussen [59] introduced independently *knot Floer homology*, which takes the form of a bi-graded, finitely generated Abelian group

$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{m,a\in\mathbb{Z}} \widehat{\mathrm{HFK}}_m(K,a),$$

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where m is called the *Maslov* (*homological*) *grading* and a is called the *Alexander grading*. The graded Euler characteristic of $\widehat{\mathrm{HFK}}(K)$ is the Alexander polynomial:

$$\Delta_K(t) = \sum_{a\in\mathbb{Z}}(\sum_{m\in\mathbb{Z}}(-1)^m \mathrm{rk}\widehat{\mathrm{HFK}}_m(K,a))t^a. \tag{1.1}$$

Conjecture 1.1 ([30, Conjecture 6.7])**.** *Let K be an L-space knot and with knot Floer homology $\widehat{\mathrm{HFK}}(K)$. Then there are only finitely many other knots whose knot Floer homology is isomorphic* (*as bi-graded groups*) *to $\widehat{\mathrm{HFK}}(K)$.*

We can reformulate Conjecture 1.1 in terms of knot genus.

Conjecture 1.2. *Given an integer $N \geq 0$, there are only finitely many L-space knots K with $g(K) = N$.*

Proof of equivalence of Conjectures 1.1 and 1.2. Assume that Conjecture 1.1 holds. Suppose for a contradiction that there are infinitely many L-space knots K with $g(K) = N$ for some non-negative integer N. If necessary, by taking mirrors, we may assume that such L-space knots are positive. Since the degree of their Alexander polynomials is bounded above by $2g(K) = 2N$, and their non-zero coefficients are ± 1 [55, Corollary 1.3], there are only finitely many Alexander polynomials. Moreover, the Alexander polynomial of an L-space knot determines its $\widehat{\mathrm{HFK}}$ [55, Theorem 1.2]; see also [41]. Thus infinitely many L-space knots share the same $\widehat{\mathrm{HFK}}$, contradicting the assumption.

To prove the converse we assume Conjecture 1.2 holds, and suppose for a contradiction that there is an L-space knot K for which there are infinitely many knots K_i $(i = 1, 2, \dots)$ with $\widehat{\mathrm{HFK}}(K_i) \cong \widehat{\mathrm{HFK}}(K)$ as bi-graded groups. By the rational surgery formula [56], any knot with $\widehat{\mathrm{HFK}}$ isomorphic to that of an L-space knot as bi-graded groups is also an L-space knot, and hence K_i $(i = 1, 2, \dots)$ is also an L-space knot. Following (1.1) we have: $\Delta_{K_1}(t) = \Delta_{K_2}(t) = \cdots$. Since L-space knots are fibered [46, 47, 18] (cf. [35]), $g(K_i) = \mathrm{deg}\Delta_{K_i}(t)/2$. This shows that infinitely many L-space knots have the same genus, contradicting the assumption. □

Of course the unknot is the only genus zero knot. The only genus one L-space knots are the trefoil and its mirror, $T_{\pm 3,2}$ [18, Theorem 1.4]. It is expected that the torus knots $T_{\pm 5,2}$ are the only L-space knots of genus two [39, Problem 13] (see also [30, Section 6.4]), but this is still open. Obviously for any non-negative integer g, there is an L-space knot $T_{p,q}$ whose genus is g. Also, the pretzel knots $P(-2, 3, 2n + 1)$ for $n \geq 0$ are all L-space knots of genus $n + 2$, but they are only hyperbolic for $n \geq 3$.

Question 1.3. *What is the minimal genus realized by hyperbolic L-space knots? In particular, are there hyperbolic L-space knots of genus 2, 3, or 4?*

In this article we examine Conjecture 1.2 for twist families of knots. The *twist family of knots* $\{K_n\}$ obtained by twisting a knot K along a disjoint unknot c is the sequence of knots that are the images of K upon $(-1/n)$–surgery on c. In the following we always assume that c neither bounds a disk disjoint from K nor is a meridian of K. Then Conjecture 1.2 for twist families of L-space knots is stated as:

Conjecture 1.4. *For any twist family of knots $\{K_n\}$ and any integer $N \ge 0$ there are only finitely many L-space knots K_n such that $g(K_n) = N$.*

What can we say about a behavior of genera of knots under twisting operation in general? If $\ell k(K, c) = 0$, then we can choose a Seifert surface S of K so that it does not intersect c. Though S may not be a minimal genus Seifert surface, it remains a Seifert surface of K_n for all integers n and thus genera of K_n are bounded above by the genus of S. For more precise statement, see Theorem 2.1, which describes an asymptotic behavior of genera of knots under twisting in general setting. However, we prove the following theorem below which results from the more general Theorem 3.1 about fibered knots in twist families together with the fact that an L-space knot or its mirror is a fibered knot supporting the tight contact structure on S^3; see Section 3.

Theorem 1.5. *Let $\{K_n\}$ be a twist family of knots obtained by twisting K along c. If $\ell k(K, c) = 0$, then K_n is an L-space knot for at most three integers n. Furthermore, if K_m and K_n are L-space knots, then $|m - n| \le 2$.*

On the other hand, for each integer $\ell > 1$ there are infinitely many twist families $\{K_n\}$ each of which contains infinitely many L-space knots with $|\ell k(K, c)| = \ell$; see [44, Theorem 1.8] or Subsection 7.2. As a direct consequence of Theorem 2.1, we have:

Theorem 1.6. *Let $\{K_n\}$ be a twist family of knots obtained by twisting K along c. If $|\ell k(K, c)| > 1$, then $g(K_n) \to \infty$ as $|n| \to \infty$. In particular, Conjecture 1.4 is true for any twist family of L-space knots with $|\ell k(K, c)| > 1$.*

Given a slope r for K, then twisting along c produces the twist family of knot-slope pairs $\{(K_n, r_n)\}$ called the *twist family of surgeries*, and the twist family of Dehn surgered manifolds $\{K_n(r_n)\}$. We call a knot-slope pair (K, r) an *L-space surgery* if $K(r)$ is an L-space. Note that if $\omega = \ell k(K, c)$, then $r_n = r_0 + n\omega^2$.

Remark 1.7. For a twist family of surgeries $\{(K_n, r_n)\}$, the genus of K_n is bounded above as a linear function of n whenever $K_n(r_n)$ is an L-space (by the relation of genus and L-space surgery slope of Ozsváth-Szabó [56]). In particular, $g(K_n) \le \frac{1}{2}(1 + |r_0 + n\omega^2|)$.

There are infinitely many twist families of surgeries each of which contains infinitely many L-space surgeries; see [44]. If $\{(K_n, r_n)\}$ contains infinitely many L-space surgeries, then of course $\{K_n\}$ contains infinitely many L-space knots. In general the converse may not be true, but we have:

Lemma 1.8. *Suppose that a twist family of knots $\{K_n\}$ contains infinitely many positive L-space knots K_{n_i} $(n_0 < n_1 < \cdots)$ of bounded genus. Then for any choice of a slope r_{n_0} for K_{n_0}, we can take a constant C so that (K_{n_i}, r_{n_i}) is an L-space surgery if $i \ge C$. In particular, the twist family of surgeries $\{(K_n, r_n)\}$ contains infinitely many L-space surgeries for any twist family of slopes r_n.*

Proof. Let G be the maximum number of genera of L-space knots K_{n_i}, and take any slope r_{n_0}. Then choose an integer $C > 0$ so that $r_{n_C} = r_{n_0} + (n_C - n_0)\omega^2 \ge 2G - 1$, where $\omega (\ne 0)$ is the linking number between K and the twisting circle c. Then if $i \ge C$, then $r_{n_i} = r_{n_0} + (n_i - n_0)\omega^2 \ge$

$r_{n_0} + (n_C - n_0) \geq 2G - 1 \geq g(K_{n_i}) - 1$. Since K_{n_i} is a positive L-space knot, following Ozsváth-Szabó [56] $K_{n_i}(r_{n_i})$ is an L-space. Hence (K_{n_i}, r_{n_i}) is an L-space surgery if $i \geq C$. □

Let us specify Conjecture 1.4 in terms of a twist family of surgeries.

Conjecture 1.9. *Let $\{(K_n, r_n)\}$ be a twist family of surgeries. Then for any integer $N \geq 0$ there are only finitely many L-space surgeries (K_n, r_n) such that $g(K_n) = N$.*

Theorems 1.5 and 1.6 verify Conjecture 1.4 and hence Conjecture 1.9 for twist families obtained by twisting K along c when $|\ell k(K, c)| \neq 1$. In case of $\ell k(K, c) = 1$, we prove:

Proposition 1.10. *Let $\{(K_n, r_n)\}$ be a twist family obtained by twisting (K, r) along an unknot c with $|\ell k(K, c)| = 1$. If this family contains infinitely many L-space surgeries, then*

(1) $\Delta_{K\cup c}(x, y) \doteq \Delta_K(x) \doteq \Delta_{K_n}(x)$ *for all* $n \in \mathbb{Z}$,
(2) $\widehat{\mathrm{HFK}}(K_n) \cong \widehat{\mathrm{HFK}}(K_N)$ *for infinitely many integers n, and in particular*
(3) $g(K_n) = g(K_N)$ *for infinitely many integers n.*

Corollary 1.11. *Let $\{(K_n, r_n)\}$ be a twist family of surgeries with $|\ell k(K, c)| = 1$. If $g(K_n) \to \infty$ as $|n| \to \infty$, then $\{(K_n, r_n)\}$ contains only finitely many L-space surgeries.*

Common examples of twist families of surgeries containing infinitely many L-space surgeries have infinitely many Seifert L-space surgeries; see [44] for such examples.

Convention 1.12. Throughout this article, we permit Seifert fibrations to have "degenerate" fibers (i.e. index zero fibers). Accordingly, a Seifert fibered space is a 3–manifold admitting a Seifert fibration with or without degenerate fibers. When we discuss surgeries, following the convention in [12], we call a knot-slope pair (K, r) a *Seifert surgery* if $K(r)$ is a Seifert fibered space in our generalized sense; if $K(r)$ is an L-space as well, then we call (K, r) a *Seifert L-space surgery.* See Section 2 in [12] for degenerate Seifert fibrations. Since connected sums of lens spaces are Seifert fibered L-spaces (in our sense), $(T_{p,q}, pq)$ is also a Seifert L-space surgery.

The next result shows finiteness of L-space surgeries in a twist family $\{(K_n, r_n)\}$ which contains at least 10 Seifert surgeries:

Theorem 1.13. *Let $\{(K_n, r_n)\}$ be a twist family of surgeries obtained by twisting (K, r) along an unknot c with $|\ell k(K, c)| = 1$. Assume that (K_n, r_n) is a Seifert surgery for at least ten integers n. Then there are only finitely many L-space surgeries in the family.*

Combining Theorems 1.5, 1.6 and 1.13, we establish Conjecture 1.9 for twist families that have at least ten Seifert surgeries:

Corollary 1.14. *Let $\{(K_n, r_n)\}$ be a twist family of surgeries in which (K_n, r_n) is an L-space surgery for infinitely many integers n and a Seifert surgery for at least ten integers n. Then $g(K_n) \to \infty$ as $|n| \to \infty$.*

1.1. **Twist families of surgeries with many Seifert fibered spaces.** In Section 4 we study when a twist family of surgeries may have a large number of Seifert surgeries without constraining the linking number $\ell k(K, c)$. In doing so, we extend the foundational work of [12] on seiferters. See Section 4 for terminology and background regarding seiferters and pseudo-seiferters. Notably, we prove the following theorem, though phrased slightly differently for its presentation here.

Theorem 4.2. Let $\{(K_n, r_n)\}$ be a twist family in S^3 obtained by twisting (K, r) along an unknot c that is neither split from K nor a meridian of K. If $K_n(r_n)$ is a Seifert fibered space for at least 10 integers n, then c is a seiferter or pseudo-seiferter for (K, r). Consequently, $K_n(r_n)$ is then a Seifert fibered space for all integers n.

Through homology calculations, Proposition 4.6 shows that $|\ell k(K, c)| \neq 1$ if c is a pseudo-seiferter for a Seifert surgery on K. However, we have not yet found a pseudo-seiferter.

Question 1.15. *Does there exist a pseudo-seiferter for a Seifert surgery on a knot in S^3?*

1.2. **Proof of Theorem 1.13 — outline.** According to Theorem 4.2, having at least 10 Seifert surgeries by hypothesis tells us that c is either a seiferter or a pseudo-seiferter for (K, r). Then, because $|\ell k(K, c)| = 1$, r is an integer (Theorem 4.5) and Proposition 4.6 implies that c must be a seiferter. In particular, (K_n, r_n) is a Seifert surgery for all n. Assume for a contradiction that our twist family contains infinitely many L-space surgeries. Using the classification of L-spaces among Seifert fibered spaces over S^2 we set the preliminary Proposition 5.3 about sequences of Seifert fibered L-spaces. Then Theorem 6.1 shows that a twist family of surgeries $\{(K_n, m_n)\}$ obtained by twisting a Seifert surgery (K, m) along a seiferter c with $|\ell k(K, c)| = 1$ contains only finitely many Seifert L-space surgeries. This completes a proof of Theorem 1.13.

1.3. **Braids and L-space knots.** We have observed that for many of the twist families containing infinitely many L-space knots that are studied in [44], the twisting circle is not only a seiferter but also a braid axis. Furthermore, L-space knots are often isotopic to closures of positive or negative braids.

In this section we investigate Conjectures 1.2 and 1.4 from a viewpoint of braid. Well known to the experts, we provide a proof of the following.

Proposition 7.1. There are only finitely many knots of each genus that are closures of positive braids.

As a direct consequence of Proposition 7.1 we have:

Corollary 1.16. *The class of L-space knots which are closures of positive and negative braids satisfy Conjecture 1.2.*

Note that the classes of closures of positive braids and L-space knots are distinct. For example, KnotInfo shows that the hyperbolic knot 10_{139} is the closure of a positive braid [9], but its Alexander polynomial indicates that it cannot be an L-space knot [55]. On the other hand, the $(3, 2)$–cable of $T_{3,2}$ is an L-space knot, but it is not a closure of any positive or negative braid, e.g. [30, 6.3].

Now let us assume that the twisting circle c is a braid axis for K, then K_n is the closure of a positive braid for $n \gg 0$ and of a negative braid for $n \ll 0$. Hence, even without Theorem 2.1, Proposition 7.1 implies Conjecture 1.4 for the twist family $\{K_n\}$.

However, a twisting circle c which is not a braid axis for K may provide a twist family $\{K_n\}$ containing infinitely many L-space knots. For instance, Example 1.2 [44] shows the pretzel knots $K_n = P(-2, 3, 2n+1)$ with the $7+4n$ surgery is a twist family with $|\ell k(K,c)| = 2$ producing Seifert fibered L-spaces for $n \geq 0$. Since K is in general not a torus knot while all 2–braids are, c is not a braid axis. Nevertheless, note that these knots are positive 3–braids when $n \geq 0$.

Torus knots K also have seiferters which are not braid axes that yield a twist families of Seifert surgeries (K_n, r_n) on non-torus knots K_n for all but finitely many n [12], and each of these twist families contain infinitely many Seifert L-space surgeries [44]. Even though these seiferters are not braid axes, in Section 7 we nevertheless demonstrate how to express these L-space knots as closures of positive or negative braids. Indeed it follows that the union of these L-space knots also satisfies Conjecture 1.2.

Question 1.17. *Is there a twist family $\{K_n\}$ containing infinitely many hyperbolic L-space knots that are not closures of positive or negative braids?*

While no example in [44] appears to give a positive answer to this question, we still expect such a twist family to exist.

Although we don't expect twist families containing infinitely many L-space knots to generically be closures of positive or negative braids, it still seems plausible that the knots should wrap coherently about the twisting circle.

Question 1.18. *If a twist family of knots $\{K_n\}$ obtained by twisting K about an unknot c contains infinitely many L-space knots, then does c bound a disk that K always intersects in the same direction?*

This behavior is observed in the examples of [44]. Furthermore, in the case that $|\ell k(K,c)| = 1$, a positive answer would imply that c is a meridian of K.

1.4. **Notation convention.**

We will use $N(*)$ to denote a tubular neighborhood of $*$ and use $\mathcal{N}(*)$ to denote the interior of $N(*)$ for national simplicity.

2. Alexander polynomials and genera of knots in twist families

In this section we will prove the general result below, which describes the behavior of the genera of knots under the twisting operation.

Theorem 2.1. *Let $\{K_n\}$ be the twist family of knots in a homology sphere obtained by twisting the knot K along an unknot c. Then one of the following occurs:*

(1) $\ell k(K,c) = 0$ *and $g(K_n)$ is constant for all but at most one n for which $g(K_n)$ may be less,*

(2) $|\ell k(K,c)| = 1$ *and* $\Delta_{K \cup c}(x,y) \doteq \Delta_K(x)$*, or*

(3) $|\ell k(K,c)| \geq 1$ *and* $g(K_n) \to \infty$ *as* $|n| \to \infty$.

Here Δ_L denotes the multivariable Alexander polynomial of the link L and $\doteq$ signifies equivalence up to multiplication by a unit in the corresponding Laurent polynomial ring.

Question 2.2. *Observe that if c is a meridian of K then $\Delta_{K\cup c}(x,y) \doteq \Delta_K(x)$ and $K_n = K$ for all n. If $|\ell k(K,c)| = 1$, $\Delta_{K\cup c}(x,y) \doteq \Delta_K(x)$, and $g(K_n) \leq N$ for some constant N, then must c be a meridian of K?*

Remark 2.3. Conclusion (3) with $|\ell k(K,c)| = 1$ does occur. For example, the two-bridge link $B(22,5)$ (which is 7^2_2 in Rolfsen's table and $L7a5$ in Thistlethwaite's table) has $|\ell k(K,c)| = 1$ and multivariable Alexander polynomial $\Delta(x,y) \doteq (x+y-1)(xy-x-y)$ [10, 36]. Since $br_y\Delta(x,y) = 2$ (the y–breadth of $\Delta(x,y)$, defined below), the proof of Theorem 2.1 shows that $g(K_n) \to \infty$ as $|n| \to \infty$.

Before proving Theorem 2.1, we prepare some notations and Lemma 2.4 below. For a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$, its *breadth* $\mathrm{br}(p(t))$ is the difference between the minimum degree and maximum degree of t in $p(t)$. (If $p(t) = 0$, then $\mathrm{br}_y(p(x,y))$ is defined to be $-\infty$.) For a Laurent polynomial $p(x,y) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, its *$y$–breadth* $\mathrm{br}_y(p(x,y))$ is the difference between the minimum degree and maximum degree of y in $p(x,y)$. We similarly define $\mathrm{br}_x(p(x,y))$.

Let K be a knot in a homology sphere M with Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ and Seifert genus $g(K)$. Then we have the inequality:

$$\mathrm{br}(\Delta_K(t)) \leq 2g(K).$$

Let $L_1 \cup L_2$ be an oriented link in a homology sphere M and E the exterior $M - \mathcal{N}(L_1 \cup L_2)$ of $L_1 \cup L_2$. The two variable Alexander polynomial of $L_1 \cup L_2$ is $\Delta_{L_1\cup L_2}(x,y)$. With the Laurent polynomial ring $\Lambda = \Lambda[x^{\pm 1}, y^{\pm 1}]$, this records the structure of $H_1(\widetilde{E})$ as a Λ module with respect to the basis $\langle [\mu_1], [\mu_2] \rangle$ of $H_1(E)$ where μ_i is an oriented meridian of L_i, $[\mu_1] \mapsto x$ and $[\mu_2] \mapsto y$, and the additive structure in $H_1(E)$ maps to the multiplicative structure in Λ (i.e. $a[\mu_1]+b[\mu_2] \mapsto x^a y^b$).

Torres [70] gives fundamental properties of the two-variable Alexander polynomial of an oriented link $L_1 \cup L_2$ with $\ell k(L_1, L_2) = \omega$ and its relation to the Alexander polynomial of a component:

$$\text{(T1)} \qquad \Delta_{L_1\cup L_2}(x,y) = x^m y^n \Delta_{L_1\cup L_2}(x^{-1}, y^{-1}) \text{ for some } m,n \in \mathbb{Z}$$

$$\text{(T2)} \qquad \Delta_{L_1\cup L_2}(t,1) \doteq \frac{t^\omega - 1}{t-1}\Delta_{L_1}(t)$$

$$\text{(T3)} \qquad \Delta_{L_1\cup L_2}(1,1) = \pm\omega$$

In the following we choose orientations of L_1 and L_2 so that $\omega = \ell k(L_1, L_2) \geq 0$.

Lemma 2.4. *Let $L_1 \cup L_2$ be an oriented link with $\ell k(L_1, L_2) = \omega \geq 0$. Then, working* mod 2*, we have*

$$\mathrm{br}_x(\Delta_{L_1\cup L_2}(x,y)) \equiv_2 \mathrm{br}_y(\Delta_{L_1\cup L_2}(x,y)) \equiv_2 \omega - 1.$$

Proof. This is an application of the Torres Formulas. If $\mathrm{br}_y(\Delta_{L_1\cup L_2}(x,y)) = n$, then by multiplying by powers of x and y we may write $\Delta_{L_1\cup L_2}(x,y) = \sum_{i=0}^n a_i(x)y^i$, where $a_0(x) \neq 0$ and $a_n(x) \neq 0$ (and possibly $0 = n$). Then by (T1) we have

$$\begin{aligned}\sum_{i=0}^{n} a_i(x)y^i &= \Delta_{L_1\cup L_2}(x,y)\\ &= x^m y^n \Delta_{L_1\cup L_2}(x^{-1},y^{-1})\\ &= x^m y^n \sum_{i=0}^{n} a_i(x^{-1})y^{-i}\\ &= x^m \sum_{i=0}^{n} a_i(x^{-1})y^{n-i}\\ &= x^m \sum_{i=0}^{n} a_{n-i}(x^{-1})y^{i}\end{aligned}$$

so that $a_i(x) = x^m a_{n-i}(x^{-1})$. Hence $a_i(1) = a_{n-i}(1)$, and therefore $\mathrm{br}_y(\Delta_{L_1\cup L_2}(x,y)) \equiv_2 \mathrm{br}(\Delta_{L_\cup L_2}(1,y))$. By (T2), $\mathrm{br}(\Delta_{L_1\cup L_2}(1,y)) = \mathrm{br}(\Delta_{L_2}(y))+\omega-1$. Since the breadth of the Alexander polynomial of a knot is always even, $\mathrm{br}(\Delta_{L_1\cup L_2}(1,y)) \equiv_2 \omega-1$. Thus $\mathrm{br}_y(\Delta_{L_1\cup L_2}(x,y)) \equiv_2 \omega-1$.

A similar proof shows $\mathrm{br}_x(\Delta_{L_1\cup L_2}(x,y)) \equiv_2 \omega-1$. □

Proof of Theorem 2.1. We choose orientations of K and c so that $\ell k(K,c) = \omega \geq 0$. When $\omega = 0$, the result follows from work of Gabai [15, Corollary 2.4]. Henceforth assume $\omega \geq 1$.

Let $E = M - \mathcal{N}(K\cup c)$ denote the exterior of $K\cup c$ where M is the homology sphere containing $K\cup c$. Then $H_1(E) = \langle[\mu_K],[\mu_c]\rangle \cong \mathbb{Z}\oplus\mathbb{Z}$ where μ_K and μ_c are oriented meridians of K and c respectively. Let λ_c be the preferred (oriented) longitude of c. Observe that $[\lambda_c] = \omega[\mu_K]$ in $H_1(E)$.

Now consider the family of links $K_n \cup c_n$ with exterior E_n obtained by $(-1/n)$–surgery on c. Observe that $E_n \cong E$ where $\mu_{K_n} \mapsto \mu_K$ and $\mu_{c_n} \mapsto \mu_c - n\lambda_c$. Thus, using that $[\mu_{c_n}] \mapsto -n\omega[\mu_K] + [\mu_c]$ in $H_1(E_n)$, we have

$$\Delta_{K_n\cup c_n}(x_n,y_n) = \Delta_{K\cup c}(x_n, x_n^{-n\omega}y_n).$$

Applying the Torres Formula (T2) and the preceding equation, we obtain:

$$(\star)\qquad \frac{t^\omega-1}{t-1}\Delta_{K_n}(t) = \Delta_{K_n\cup c_n}(t,1) = \Delta_{K\cup c}(t,t^{-n\omega})$$

Since $\omega \geq 1$, we have

$$(\star\star)\qquad 2g(K_n) \geq \mathrm{br}(\Delta_{K_n}(t)) = \mathrm{br}(\Delta_{K\cup c}(t,t^{-n\omega})) - (\omega-1)$$

Thus the genus of K_n will eventually increase with $|n|$ provided that we have $\mathrm{br}_y(\Delta_{K\cup c}(x,y)) > 0$.

Since c is the unknot, $\Delta_c(y) = 1$. Therefore

$$\Delta_{K\cup c}(1,y) = \frac{y^\omega-1}{y-1}\Delta_c(y) = \frac{y^\omega-1}{y-1}$$

and thus $\Delta_{K\cup c}(x,y)$ has positive y–breadth when $\omega \geq 2$. Hence conclusion (3) holds when $\omega \geq 2$.

If $\omega = 1$, then $\Delta_{K\cup c}(1,y) = \Delta_c(y) = 1$, which implies that $\Delta_{K\cup c}(x,y) \neq 0$. However, if $\mathrm{br}_y(\Delta_{K\cup c}(x,y)) = 0$, then $\Delta_{K\cup c}(x,y)$ is expressed as $f(x)y^k$ for some polynomial $f(x)$ and integer k, and hence $\Delta_{K\cup c}(x,y) \doteq f(x)$. Therefore $(\star)$ implies that $\Delta_{K\cup c}(x,y) \doteq \Delta_{K\cup c}(x,1) = \Delta_K(x)$

and moreover that $\Delta_{K_n}(x) = \Delta_{K_n \cup c_n}(x,1) = \Delta_{K\cup c}(x, y^{-nw}) \doteq \Delta_{K\cup c}(x,1) = \Delta_K(x)$ for all $n \in \mathbb{Z}$. Thus if $\omega = 1$, then either $\Delta_{K\cup c}(x,y) \doteq \Delta_K(x) \doteq \Delta_{K_n}(x)$ for all $n \in \mathbb{Z}$ and conclusion (2) holds or $\mathrm{br}_y(\Delta_{K\cup c}(x,y)) > 0$ and conclusion (3) holds. □

Proposition 1.10. Let $\{(K_n, r_n)\}$ be a twist family obtained by twisting (K, r) along an unknot c with $|\ell k(K,c)| = 1$. If this family contains infinitely many L-space surgeries, then

1. (1) $\Delta_{K\cup c}(x,y) \doteq \Delta_K(x) \doteq \Delta_{K_n}(x)$ for all $n \in \mathbb{Z}$,
2. (2) $\widehat{\mathrm{HFK}}(K_n) \cong \widehat{\mathrm{HFK}}(K_N)$ for infinitely many integers n, and in particular
3. (3) $g(K_n) = g(K_N)$ for infinitely many integers n.

Remark 2.5. Of course, as in Question 2.2, we know of no examples of twist families that satisfy all the hypotheses of Proposition 1.10 for which c is not a meridian of K.

Proof. Note that the assertion of the proposition holds for $\{(K_n, r_n)\}$ if and only if that holds for the family $\{(K^*_{-n}, -r_n)\}$ obtained by taking mirrors. So we may assume that there is an integer $N > 0$ such that (K_n, r_n) is an L-space surgery for infinitely many $n \geq N$. In the following we choose orientations of K and c so that $\ell k(K,c) = 1$. Then, since $r_n = r_0 + n$, by increasing N if necessary we may assume $r_n > 0$ so that K_n is a positive L-space knot for infinitely many $n \geq N$.

Since K_n is a positive L-space knot, then $r_n \geq 2g(K_n) - 1$ [56]. Then equation ($\star\star$) above yields

$$r_0 + n\omega^2 \geq \mathrm{br}(\Delta_{K\cup c}(t, t^{-n\omega})) - \omega.$$

Recall that, as in the proof of Lemma 2.4, if $\mathrm{br}_y(\Delta_{K\cup c}(x,y)) = \ell$, then we may write $\Delta_{K\cup c}(x,y) = \sum_{i=0}^{\ell} a_i(x) y^i$ where the $a_i(x)$ are polynomials such that $a_0(x) \neq 0$, $a_\ell(x) \neq 0$, and $x^k a_i(x^{-1}) = a_{\ell-i}(x)$ for all i for some integer k.

Then $\Delta_{K\cup c}(t, t^{-n\omega}) = a_0(t) + \cdots + a_\ell(t) t^{-n\omega\ell} \doteq a_0(t) t^{n\omega\ell} + \cdots + a_\ell(t)$ and for sufficiently large $n (\geq N)$, $\mathrm{br}(\Delta_{K\cup c}(t, t^{-n\omega})) = n\omega\ell + C$ where C is a constant $\deg a_0(x) - \deg a_\ell(x)$ so that the inequality above becomes

$$n\omega(\omega - \ell) \geq C - \omega - r_0$$

For this inequality to be true for sufficiently large $n \geq N$, we must have $\omega \geq \ell$. That is, $\ell k(K,c) \geq \mathrm{br}_y(\Delta_{K\cup c}(x,y)) \geq 0$. In particular, since ω and ℓ do not have the same parity by Lemma 2.4, if $\omega = \ell k(K,c) = 1$ then $\ell = \mathrm{br}_y(\Delta_{K\cup c}(x,y)) = 0$. This implies $\Delta_{K\cup c}(x,y) \doteq \Delta_K(x)$ and thus, as in Theorem 2.1, $\Delta_K(x) \doteq \Delta_{K_n}(x)$ for all $n \in \mathbb{Z}$.

Since Alexander polynomials of L-space knots determine their $\widehat{\mathrm{HFK}}$, this then implies $\widehat{\mathrm{HFK}}(K_n) = \widehat{\mathrm{HFK}}(K_N)$ for infinitely many $n \geq N$. □

3. L-space knots in twist families with linking number zero

As shown in Theorem 2.1, twisting K along an unknotted circle c with $\ell k(K,c) = 0$, we obtain an infinite family of knots of bounded genus. If this family contains infinitely many L-space knots, Conjecture 1.2 turns out to be not true. However Theorem 1.5 below, which follows from Theorem 3.1 and the fact that an L-space knot or its mirror is a tight fibered knot, excludes this possibility. (For convenience, we say a fibered knot whose associated open book decomposition supports the positive tight contact structure on S^3 is a *tight fibered knot.*)

Theorem 1.5. Let $\{K_n\}$ be a twist family of knots obtained by twisting K along c. If $\ell k(K, c) = 0$, then K_n is an L-space knot for at most three integers n. Furthermore, if K_m and K_n are L-space knots, then $|m - n| \leq 2$.

Proof. By Ni [46, 47] (cf. [18, 35]), if K is an L-space knot, then K is a fibered knot. If K is an L-space knot with a positive L-space surgery, then $g(K) = \tau(K)$ [55] (see also [29, Corollary 1.4]) and the open book decomposition associated to K supports the (positive) tight contact structure on S^3 [29, Proposition 2.1]. That is, K is a tight fibered knot. Similarly, if K is an L-space knot with a negative L-space surgery, then the mirror of K is a tight fibered knot. The result now follows from Theorem 3.1 below. □

Theorem 3.1. *Let $\{K_n\}$ be a twist family of knots obtained by twisting K along c. If $\ell k(K, c) = 0$, then K_n or its mirror is a tight fibered knot for at most three integers n. Furthermore, if K_m and K_n are two such knots, then $|m - n| \leq 2$.*

Proof. If $\{K_n\}$ contains no fibered knots, then there is nothing to prove. So we may assume, if necessary by a reparametrization, that $K = K_0$ is a fibered knot. It follows from [15, Corollary 2.4] that K has a Seifert surface $F \subset E(K) = S^3 - \mathcal{N}(K)$ which is disjoint from c so that $g(K_n) \leq g(F)$ with equality for all but at most one integer n, say n_0. (Cf. Theorem 2.1(1).) In particular, the image of F under $(-\frac{1}{n})$–surgery on c gives a minimal genus Seifert surface for K_n in those cases of equality.

Case I. $g(K) = g(F)$, i.e. F is a fiber surface of K.

Since F is a fiber surface, by cutting the exterior $E(K)$ along F one obtains a product manifold $F \times [0, 1]$. Assume that K_n or its mirror is also a fibered knot for some integer $n \neq 0, n_0$. Thus K_n is a fibered knot with $g(K_n) = g(F)$, and since a fiber surface for a fibered knot is unique up to isotopy (e.g. [13, Lemma 5.1] or [69]), F becomes a fiber surface F_n of K_n after $(-\frac{1}{n})$–surgery on c. Hence $(-\frac{1}{n})$–surgery on c takes the exterior of $K \cup F$ to the exterior of $K_n \cup F_n$; i.e. this is a cosmetic surgery of $F \times [0, 1]$ such that $F \times \partial[0, 1]$ is preserved. Then Ni [49, Theorem 1.1] shows that c may be isotoped so that in the projection $\pi : F \times [0, 1] \to F$, either (i) the projection of c has no crossings, or (ii) the projection of c has just one crossing.

The immersed annulus $\pi^{-1}(\pi(c))$ intersects $\partial N(c)$ in two longitudes and a meridian for each crossing of $\pi(c)$. The slope of these longitudes is referred to as the *blackboard framing*.

In situation (i), assume that K_n is a fibered knot as above for two integers $n = n_1, n_2$ other than 0 and n_0. Then we have:

Claim 3.2. *The blackboard framing is the preferred longitude of c.*

Proof. Let γ be the blackboard framing of c. Then $\gamma = x\mu + \lambda$ for some integer x, where (μ, λ) is a preferred meridian longitude pair of c in S^3. By [49, Theorem 1.1] the distance between the surgery slope $-\frac{1}{n}$ and γ is one. Thus $|1 + nx| = 1$ for the nonzero integers $n = n_1, n_2$. This then implies $x = 0$, i.e. $\gamma = \lambda$. □

Now isotope c into the fiber surface F for K; we continue to use the same symbol c to denote the isotoped one. Then c is essential in F, for otherwise, c bounds a disk disjoint from K, contradicting the assumption. Since $c \subset F$ is unknotted in S^3 and its framing by F is its preferred longitude (Claim 3.2), c is a "twisting loop" as in [74, Definition 2.1]. Then it follows from [74, Theorem 1.1] that any contact structure supported by the open book with page F will be overtwisted. Similarly, since the mirror of F also contains a twisting loop, the mirror of c, any contact structure it supports will also be overtwisted. (Indeed, one may show that in the supported contact structures the twisting loop can be isotoped to a Legendrian unknot that bounds an overtwisted disk.) In particular, c is a curve in F along which one may do a "Stallings twist" and so it further follows that for every member of the twist family $\{K_n\}$ the knot K_n is fibered with fiber F_n in which c continues to be a twisting loop.

Hence if K_n is fibered for at least n_1 and n_2 $(n_1, n_2 \neq 0, n_0)$, then for every integer n the knot K_n is fibered and neither K_n nor its mirror is a tight fibered knot.

Therefore there are at most two non-zero integers n_0 and n_1 such that K_0, K_{n_0} and K_{n_1}, or their mirrors are tight fibered knots in the family $\{K_n\}$.

Let us turn to situation (ii). Recall that $K = K_0$ is assumed to be a fibered knot. First we observe that F is incompressible in $E(K_n)$ for all integers n. It is sufficient to show that $F = F \times \{0\}$ remains incompressible in the resulting 3–manifold X_n obtained from $F \times [0,1]$ after $(-\frac{1}{n})$–surgery on c for all integers n. Assume for a contradiction that $F = F \times \{0\}$ compresses in X_n after $(-\frac{1}{n})$–surgery on c for some n. Then [49, Theorem 1.5] or [63, Theorem 0.1] (see also [48, Theorem 1.4]) implies that the projection of c has no crossings contradicting the hypothesis of situation (ii). Next we show that there is at most one non-zero integer n such that K_n is also fibered. If K_n is also a fibered knot for $n \neq 0$, then since F is incompressible in $E(K_n)$ as observed above, $g(K_n) = g(F)$ and the fiber F of K becomes a fiber surface F_n for K_n after $(-\frac{1}{n})$–surgery on c [13, Lemma 5.1] ([69]). Thus $(-\frac{1}{n})$–surgery on c is also a cosmetic surgery of $F \times [0,1]$. Since the cosmetic surgery slope is exactly the blackboard framing [49, Theorem 1.1], this non-zero integer n is unique.

Therefore there is at most one non-zero integer n_1 such that both K_0 and K_{n_1} are fibered knots in the family $\{K_n\}$.

Finally let us prove that if K_m and K_n or their mirrors are tight fibered knots, then $|m-n| \leq 2$. We reparametrize the family $\{K_n\}$ so that $K = K_0$ or its mirror is a tight fibered knot as above and then take a closer look at the values n_0 and n_1.

First we consider the situation (i) in which the projection of c in $F \times I$ to F has no crossings.

Recall that $(-\frac{1}{n})$–surgery on c compresses F for at most one integer n which we denote as n_0 should it exist; such a surgery is a ∂–reducing surgery in $F \times I$. Since the blackboard framing of c is the only slope of a Dehn surgery on c in which F compresses (see [49, Theorem 1.5] or [63, Theorem 0.1]), the slope $-\frac{1}{n_0}$ must be the blackboard framing. Hence if $(-\frac{1}{n_0})$–surgery on c compresses F, then $-\frac{1}{n_0}$ is an integer, and thus $n_0 = \pm 1$.

Recall that $(-\frac{1}{n_1})$–surgery is a cosmetic surgery of $F \times [0,1]$ and $n_1 (\neq 0)$ satisfies $|1 + n_1 x| = 1$ for some integer x. If $x = 0$, then the blackboard framing is a preferred longitude of c (so there is no n_0 for which $(-\frac{1}{n_0})$–surgery on c compresses F) and we arrive at a contradiction in which every member of $\{K_n\}$ and its mirror fails to be a tight fibered knot as above. So $x \neq 0$ and the equality implies $n_1 = \pm 1, \pm 2$. Summarizing, we see that if K_n or its mirror is tight fibered knot, then $n \in \{-2, -1, 0, 1, 2\}$. If none of K_{-2}, K_2, or their mirrors are tight fibered knots, then K_n or its mirror can be a tight fibered knot for at most three integers $n = -1, 0, 1$ providing the desired result. Suppose that K_2 or its mirror is a tight fibered knot. Since $-\frac{1}{2}$ is not the slope of a ∂–reducing surgery, $g(K_2) = g(F)$ and we can replace K_2 with $K = K_0$ by reparametrization and apply the same argument to conclude that if K_n or its mirror is a tight fibered knot, then $n \in \{0, 1, 2, 3, 4\}$. Taking the previous restriction, we have only three integers $n = 0, 1, 2$ for which K_n or its mirror can be a tight fibered knot. In the case where K_{-2} or its mirror is a tight fibered knot, a similar argument shows that K_n or its mirror can be a tight fibered knot for at most three integers $n = -2, -1, 0$. It follows that if K_m and K_n or their mirrors are tight fibered knots (without any reparametrization), then $|m - n| \leq 2$.

In the situation (ii), it is sufficient to look at the value n_1. Recall that $(-\frac{1}{n_1})$ is the blackboard framing, which is integral. Hence $n_1 = \pm 1$, and K_n or its mirror can be a tight fibered knot for at most two integers $0, -1$ or at most two integers $0, 1$.

Case II. $g(K) < g(F)$, i.e. F is not a fiber surface of K and the fiber surface of K cannot be made disjoint from c. Then it turns out that $g(K_n) = g(F)$ for any integer $n \neq 0$ as we mentioned above. Assume that K_{n_0} is a fibered knot for some $n_0 \neq 0$. Apply the argument in Case I to K_{n_0} instead of $K = K_0$, we see that there are at most three knots including K, K_{n_0} that themselves or their mirrors are tight fibered knots, and if K_m and K_n are two such knots (without any reparametrization), then $|m - n| \leq 2$. □

Example 3.3. Let $K \cup c$ be the Whitehead link depicted in Figure 3.1. Then the linking number between K and c is zero and the twist family $\{K_n\}$ contains exactly two L-space knots $K = K_0$ and K_1. Even though K_{-1} is also fibered, both K_{-1} and its mirror support overtwisted contact structures. Hence K_{-1} cannot be an L-space knot.

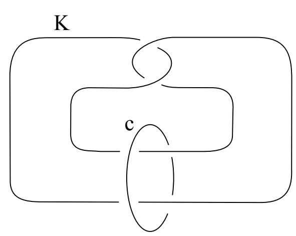

FIGURE 3.1. The linking number between K and c is zero; $K = K_0$ is a trivial knot, K_1 is a trefoil knot, and K_{-1} is the figure eight knot.

4. Twist families of Seifert surgeries; seiferters and pseudo-seiferters

Here review and extend the foundations of [12]. Recall that the term Seifert surgery means a knot-slope pair (K, r) in S^3 such that the result $K(r)$ of r–Dehn surgery on K is a manifold that admits a Seifert fibration, possibly with degenerate fibers. If an unknot c in the exterior of K becomes isotopic to a fiber in a Seifert fibration of $K(r)$, then c is called a *seiferter*: twisting the Seifert surgery (K, r) along c produces a 1–parameter twist family $\{(K_n, r_n)\}$ of Seifert surgeries. Typically it is assumed that any disk bounded by c is intersected by K at least twice; otherwise c is either split from K or a meridian of K, sometimes called an "irrelevant" seiferter.

Let c be a seiferter for a Seifert surgery (K, r). The exterior of c is a solid torus $V = S^3 - \mathcal{N}(c)$ containing K so that the manifold $V(K; r)$ resulting from r–Dehn surgery on K in V has a Seifert fibration. The main result of [43] (see also [12, Theorem 2.2]) is that either $r \in \mathbb{Z}$ or K is a torus knot in V or a cable of a torus knot in V. Hence the situation when c is a seiferter for a Seifert surgery (K, r) with $r \notin \mathbb{Z}$ is well understood. Thus [12] focuses upon integral Seifert surgeries (K, m) where $m \in \mathbb{Z}$. (In reference to notation for surgery slopes, we always take $m \in \mathbb{Z}$ while in general $r \in \mathbb{Q}$.) Theorems 3.2 and 3.19 of [12] classify seiferters for integral Seifert surgeries (K, m).

One generalization of a seiferter is that of a *pseudo-seiferter*, cf. [44, Definition 8.4]. Given a Seifert surgery (K, r), an unknot c in the exterior of K is a pseudo-seiferter if c is isotopic to the cable of a fiber in some Seifert fibration of $K(r)$ and the preferred longitude λ of c in S^3 becomes the cabling slope of c in $K(r)$. In particular, the manifold $V(K; r)$ is a graph manifold that is the union along a torus of a Seifert fibered space X and a cable space W; the slope $\lambda \subset \partial V \subset \partial W$ is the cabling slope of the cable space.

In the definition of a pseudo-seiferter, the condition that λ becomes the cabling slope of W is precisely what's needed for $W_n = W \cup_{-\frac{1}{n}} N(c)$, the filling corresponding to $(-1/n)$–surgery on c, to be a solid torus. This allows the Seifert fibration of X to extend to a Seifert fibration of $K_n(r_n)$. Hence again, twisting the Seifert surgery (K, r) along c produces a 1–parameter family $\{(K_n, r_n)\}$ of Seifert surgeries.

In the following two subsections we show that if a twisting a surgery (K, r) along an unknot c produces family $\{(K_n, r_n)\}$ with ten Seifert surgeries, then

- (Theorem 4.2) c is either a seiferter or a pseudo-seiferter and so each surgery (K_n, r_n) is a Seifert surgery; and thence
- (Theorem 4.5) either $r \in \mathbb{Z}$ or c is a seiferter and K is either a torus knot or a cable of a torus knot in $V = S^3 - \mathcal{N}(c)$.

Remark 4.1. The Seifert fibrations in this article are permitted to have degenerate exceptional fibers. Do note, however, that a Seifert fibered space obtained by surgery on a knot in S^3 cannot have more than one degenerate fiber unless the knot is trivial and the surgery is the 0–slope. See Proposition 2.8 [12].

4.1. Seifert surgeries in twist families.

Let $\{(K_n, r_n)\}$ be a twist family in S^3 obtained by twisting (K, r) along an unknot c. Recall that c neither bounds a disk disjoint from K nor is a meridian of K. Let us write:

$$\mathcal{S} = \{n \in \mathbb{Z} \mid K_n(r_n) \text{ is a (possibly degenerate) Seifert fibered space}\}.$$

If $\mathcal{S} \neq \emptyset$, by reparametrization we assume $K(r)$ is a (possibly degenerate) Seifert fibered space.

Theorem 4.2. *If $|\mathcal{S}| > 9$, then c is either a seiferter or a pseudo-seiferter for (K, r) and $\mathcal{S} = \mathbb{Z}$.*

Proof. By the "inheritance" property [12, Propositio 2.6], c is a seiferter (or a pseudo-seiferter) for (K, r) if and only if (K_n, r_n) is a Seifert surgery for which c remains a seiferter (or a pseudo-seiferter) for any $n \in \mathbb{Z}$. So showing that c is a seiferter or pseudo-seiferter implies that $\mathcal{S} = \mathbb{Z}$. Hence we assume that $|\mathcal{S}| > 9$ and aim to show that c is a seiferter or pseudo-seiferter.

Let $V = S^3 - \mathcal{N}(c)$ be the solid torus exterior of c which contains the knot K, and let $V(K;r) = K(r) - \mathcal{N}(c)$. Use the preferred meridian-longitude slopes μ and λ for $\partial N(c)$ to parametrize slopes in both ∂V and $\partial V(K;r)$. Then observe that $K_n(r_n)$ is the result of filling $V(K;r)$ along the slope $\mu - n\lambda$, i.e. $K_n(r_n) = V(K;r) \cup_{-\frac{1}{n}} N(c)$.

Scharlemann's [62] strengthening of Gabai's work on surgeries on knots in solid tori [17] shows that either

(1) $V(K;r)$ is a solid torus (and so either K is a 0–bridge braid in V or K is a 1–bridge braid in V, see also [3]);
(2) $V(K;r) \cong W \# L(p,q)$, K is a (p,q)–cable knot in V, $p \geq 2$, and r is the cabling slope; or
(3) $V(K;r)$ is irreducible and ∂–irreducible.

Since $K_n(r_n) = V(K;r) \cup_{-\frac{1}{n}} N(c)$ is a Seifert fibered space for more than nine integers n, $V(K;r)$ is not hyperbolic [38, Theorem 1.2], cf. [1].[1] Therefore $V(K;r)$ is either reducible, ∂–reducible, Seifert fibered (with non-degenerate Seifert fibration), or toroidal. If $V(K;r)$ is Seifert fibered, then c is a seiferter; so let us assume $V(K;r)$ is not Seifert fibered. Thus either $V(K;r)$ is reducible and as in the second case above or $V(K;r)$ is toroidal and as in the third case above.

Case I: $V(K;r)$ is reducible.
If $V(K;r)$ is reducible, it has a lens space summand $L(p,q)$ with $p \geq 2$, K is a cabled knot in V, and r is the cabling slope. Say K is a cable of a knot J in V; J is not a core of V because $V(K;r)$ is not Seifert fibered. Hence K_n is a cable of the knot J_n obtained by twisting J along c, and r_n is the cabling slope. Since the unknot c does not bound a disk that is either disjoint from J or intersected by J just once, J_n becomes a trivial knot in S^3 for at most two integers n [17] (cf. [37, 42]). In the following we take $n \in \mathcal{S}$ so that J_n is not a trivial knot in S^3. So assuming $K_n(r_n)$ is Seifert fibered, either it is irreducible and thus just the lens space $L(p,q)$ or it is reducible and either $L(2,1)\#L(2,1)$ with no degenerate fibers or a connected sum of two lens spaces with one degenerate fiber (cf. [12, Proposition 2.8]). For homological reasons, $K_n(r_n)$ cannot be $L(2,1)\#L(2,1)$.

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[1] Indeed, Thurston's Hyperbolic Dehn Surgery Theorem [67, 68, 2, 57, 7] implies that if $K_n(r_n)$ is not hyperbolic for infinitely many n, then $V(K;r)$ is not hyperbolic. However explicit bounds have been obtained on the number of non-hyperbolic fillings a hyperbolic manifold may have. While [38] determines the optimal bound for hyperbolic manifolds with one cusp, as suggested by [1] it is conceivable fewer Seifert fibered fillings are needed for our particular situation. Our argument also requires a bound for filling multiple cusps, in which case the distance between two non-hyperbolic filling is less that or equal to 8; see [21, Table 2.1].

Assume that $K_n(r_n)$ is a lens space for some $n \in \mathcal{S}$. Then we appeal to the classification of lens space surgeries on satellite knots [5, Theorem 1]. Since J_n is non-trivial in S^3, then it is a torus knot. Therefore K_n is a cable of this torus knot in S^3 and r_n is an integral slope intersecting the cabling slope once. Yet since a non-trivial knot cannot be expressed as a non-trivial cable of J_n in more than one way, r_n cannot also be a cabling slope. This is a contradiction.

Hence $K_n(r_n)$ is a connected sum of lens spaces for $n \in \mathcal{S}$. Greene showed that K_n must be the cable of a torus knot where the surgery is along the cabling slope [27]. Since we have chosen n so that J_n is nontrivial, this implies that J_n is a nontrivial torus knot in S^3 for each $n \in \mathcal{S}$. Let us determine the position of J in V.

Claim 4.3. *J is a 0–bridge braid in V. In particular, K is a cable of a 0–bridge braid in V.*

Proof. If $V - \mathcal{N}(J)$ is Seifert fibered, then it is a cable space and we have the desired conclusion. So we exclude the remaining possibilities of $V - \mathcal{N}(J)$ being hyperbolic, reducible, or toroidal. If hyperbolic, following [25, Corollary 1.2], there are at most four integers n such that J_n is a nontrivial torus knot, a contradiction. If reducible, then J must be contained in a ball in V; thus c bounds a disk disjoint from K, a contradiction. Thus we assume that $V - \mathcal{N}(J)$ is toroidal. Let $\mathcal{T}$ be a family of tori that gives the torus decomposition of $V - \mathcal{N}(J)$ in the sense of Jaco-Shalen [32] and Johannson [34] [2]. See also [28]. Let X be the decomposing piece which contains ∂V; $X \neq V - \mathcal{N}(J)$. If $X \cup_{-\frac{1}{n}} N(c)$ is ∂-irreducible for some $n \in \mathcal{S}$, then a component T of $\partial(X \cup_{-\frac{1}{n}} N(c))$ is an essential torus in the torus knot space $S^3 - \mathcal{N}(J_n) = (V - \mathcal{N}(J)) \cup_{-\frac{1}{n}} N(c)$, a contradiction. Thus $X \cup_{-\frac{1}{n}} N(c)$ is ∂-reducible for any $n \in \mathcal{S}$. Hence [11, Theorem 2.0.1] shows that X is a cable space and the distance between the slope $-\frac{1}{n}$ and that of the fiber slope of X on ∂V is at most one. This then implies that the fiber slope coincides with the longitudinal slope λ of c in ∂V. Let $V_X \subset V$ be the solid torus bounded by $T = \partial X - \partial V$ so that $V = X \cup_T V_X$ and V_X contains J and K. Then since V is a solid torus, the meridian of V_X must intersect a regular fiber of X in T just once. Therefore the core of V_X is isotopic in V to λ. In particular, there is a meridional disk of V disjoint from V_X. Hence c bounds a disk disjoint from K, a contradiction. □

Thus K_n is a cable of a torus knot J_n and c is a basic seiferter for the companion torus knot J_n. It follows from [12, Proposition 8.7] that c is a seiferter for $K_n(m_n)$, hence for $K(r)$.

Case II: $V(K;r)$ is toroidal, irreducible, ∂–irreducible, and not Seifert fibered.
Since $V(K;r)$ is irreducible, except for at most two integers n, $K_n(r_n)$ is irreducible and hence not a connected sum of lens spaces [23, Theorem 1.2]. Let us observe that any essential torus must separate $V(K;r)$. If $V(K;r)$ contains a non-separating essential torus T, then there would be a non-separating torus in $K_k(r_k)$ for all integers k, and we must have $r_k = 0$ for homological reason. If T is compressible in $K_k(r_k)$, then compress T to obtain a non-separating 2–sphere. Then $K_k(r_k) = S^1 \times S^2$ and K_k is the trivial knot O. Suppose that T is essential in $K_k(r_k)$. Since the base surface of $K_k(r_k)$ is S^2 or $\mathbb{R}P^2$, we may assume T is horizontal (intersecting fibers transversely) [28, Proposition 1.11]. (In an orientable Seifert fibered space over $\mathbb{RP}^2$, a non-separating vertical

[2] We say that a family of tori $\mathcal{T}$ gives a torus decomposition of an irreducible 3–manifold M, if each member of $\mathcal{T}$ is an essential torus and each decomposing piece (i.e. component) obtained by cutting M along all of these tori is Seifert fibered or hyperbolic and no proper subfamily of $\mathcal{T}$ has this property.

surface is a Klein bottle.) Then cutting $K_k(r_k)$ along T, we obtain $T \times [0,1]$, i.e. $K_k(r_k)$ is a torus bundle over S^1. Following [16, Corollaries 8.3 and 8.19] we see that K_k is a genus one fibered knot, i.e. a trefoil knot or the figure-eight knot. Since 0–surgery on the figure-eight knot does not result in a Seifert fibered space, K_k is a trefoil knot for any integer k. It follows that $\{K_k\}_{k\in\mathbb{Z}} \subset \{O, T_{-3,2}, T_{3,2}\}$. This then implies that c bounds a disk intersecting K at most once [37], a contradiction.

Let $\mathcal{T}$ be a family of essential tori in $V(K;r)$ which gives a torus decomposition of $V(K;r)$. By assumption $\mathcal{T}$ is non-empty and, as shown above, consists of separating tori.

Let X be the decomposing piece which contains ∂V. If X is hyperbolic, then referring Table 2.1 in [21] which summarizes [20, 23, 24, 26, 51, 58, 62, 72, 73], we see that there are at most nine integers k such that $X \cup_{-\frac{1}{k}} N(c)$ is not hyperbolic. Since $|\mathcal{S}| > 9$ we have an integer $n \in \mathcal{S}$ for which $X \cup_{-\frac{1}{n}} N(c)$ is also hyperbolic. But then $\mathcal{T}$ gives a torus decomposition for $K_n(r_n)$ as well, contradicting that $K_n(r_n)$ is Seifert fibered. Hence X admits a Seifert fibration.

Let T be a component of $\partial X - \partial V$. We now divide into two cases depending on whether, in $X \cup_{-\frac{1}{n}} N(c)$, (i) T is compressible for at most two integers $n \in \mathcal{S}$, or (ii) T is compressible for more than two integers $n \in \mathcal{S}$. In the following we show that the first case does not occur and the second case leads us to conclude that c is a pseudo-seiferter for (K, r).

(i) Suppose that T is compressible in $X \cup_{-\frac{1}{n}} N(c)$ for at most two integers $n \in \mathcal{S}$. We can choose $n \in \mathcal{S}$ so that T is incompressible in $X \cup_{-\frac{1}{n}} N(c)$. Since X admits a Seifert fibration, any Seifert fibration of X extends to one on $X \cup_{-\frac{1}{n}} N(c)$. Note that since T is incompressible in $X \cup_{-\frac{1}{n}} N(c)$, the extended Seifert fibration is non-degenerate [12, Lemma 2.7] and unique except when $X \cup_{-\frac{1}{n}} N(c)$ is $S^1 \times S^1 \times [0,1]$ or the twisted I–bundle over the Klein bottle [31, VI.18]. As Claim 4.4 below shows, except for at most two integers n, these exceptional cases cannot occur. Hence we can take $n \in \mathcal{S}$ so that the Seifert fibration of $X \cup_{-\frac{1}{n}} N(c)$ is the extension of a Seifert fibration of X. Since $K_n(r_n)$ admits a Seifert fibration, the Seifert fibration now on X must be compatible with that of the next decomposing pieces along the tori $\partial X - \partial V$. Thus $V(K;r)$ was already Seifert fibered, contradicting our original assumption for Case II.

Claim 4.4. *There are at most two integers $n \in \mathcal{S}$ such that $X \cup_{-\frac{1}{n}} N(c)$ is $S^1 \times S^1 \times [0,1]$ or a twisted I–bundle over the Klein bottle.*

Proof. First assume that $X \cup_{-\frac{1}{n}} N(c)$ is a twisted I–bundle over the Klein bottle. Then it has exactly two Seifert fibrations: a Seifert fibration over the disk with two exceptional fibers of indices 2 and a Seifert fibration over the Möbius band with no exceptional fibers [71, Lemma 1.1]. Accordingly X is a cable space with an exceptional fiber of index 2 or a circle bundle over the once punctured Möbius band. In either case we assume that the regular fiber has slope $\frac{x}{y}$ on $\partial V = \partial N(c)$ for some relatively prime integers x, y. In the former, since the distance between the slope $-\frac{1}{n}$ and the fiber slope $\frac{x}{y}$ is $|nx + y|$, which can be two at most two values of n. In the latter, the distance between the slope $-\frac{1}{n}$ and the fiber slope is $|nx + y|$ can be one for infinitely many integers if $x = 0, y = \pm 1$, i.e. the regular fiber is a longitude of c. Since $K_n(r_n)$ is Seifert fibered, the extended Seifert fibration of $X \cup_{-\frac{1}{n}} N(c)$ (over the Möbius band) or the alternative Seifert fibration of $X \cup_{-\frac{1}{n}} N(c)$ over the disk is compatible with the next decomposing piece. If we have the first situation, then $V(K;r)$ is Seifert fibered, contradicting the assumption. So we have the second situation, but the slope of a regular fiber of the Seifert fibration over the disk varies according to n, and hence there is at most one integer n such that the Seifert fibration of $X \cup_{-\frac{1}{n}} N(c)$ (over the disk) coincides with that of the next decomposing piece. Thus there are at most two integers n such that $X \cup_{-\frac{1}{n}} N(c)$ is a twisted I–bundle over the Klein bottle.

Suppose next that $X \cup_{-\frac{1}{n}} N(c)$ is $S^1 \times S^1 \times [0,1]$. This can happen only when $\partial X - \partial V$ consists of two components T_0 and T_1; each T_i bounds a ∂-irreducible 3–manifold Y_i in $V(K;r)$. Note also that since $K_n(r_n)$ is a Seifert fibered space, both Y_0 and Y_1 are Seifert fibered spaces each of which has a unique Seifert fibration (up to isotopy) if Y_i is not a twisted I–bundle over the Klein bottle; if Y_i is a twisted I–bundle over the Klein bottle, it has exactly two Seifert fibrations as we mentioned above.

Assume that $X \cup_{-\frac{1}{n_j}} N(c) \cong S^1 \times S^1 \times [0,1]$ for $j = 0,1,2$. Viewing $c_0^* \subset X \cup_{-\frac{1}{n_0}} N(c) = S^1 \times S^1 \times [0,1]$ with $(-\frac{1}{n_0})$–slope a meridian of c_0^*, it admits a nontrivial surgery yielding $(S^1 \times S^1 \times [0,1])(c_0^*; -\frac{1}{n_j}) \cong S^1 \times S^1 \times [0,1]$ for $j = 1,2$. (Here we continue to use $-\frac{1}{n_j}$ to denote the slope of c_0^* which corresponds with the slope $-\frac{1}{n_j}$ of c.) Since $S^1 \times S^1 \times [0,1] - \mathcal{N}(c_0^*)(= X)$ is irreducible, Ni [49, Theorem 1.1] shows that $c_0^* \subset S^1 \times S^1 \times \{\frac{1}{2}\}$. Taking an obvious vertical annulus A_i which connects $\partial N(c_0^*)$ and $T_i = S^1 \times S^1 \times \{i\}$, $(-\frac{1}{n_j})$–surgery on c_0^* corresponds to some annulus twist along A_1.

Since $K_{n_0}(r_{n_0}) = Y_0 \cup (X \cup_{-\frac{1}{n_0}} N(c)) \cup Y_1 = Y_0 \cup (S^1 \times S^1 \times [0,1]) \cup Y_1$ is Seifert fibered, the torus ∂Y_0 is either vertical (consisting of fibers) or horizontal (intersecting fibers transversely) in $K_{n_0}(r_{n_0})$ [28, Proposition 1.11]. If the latter case happens, then both Y_0 and Y_1 are twisted I–bundle over the Klein bottle and $K_{n_0}(r_{n_0})$ has a Seifert fibration over the Klein bottle. This is impossible for homological reasons. So ∂Y_0 is vertical and the Seifert fibration of $X \cup_{-\frac{1}{n_0}} N(c) = S^1 \times S^1 \times [0,1]$ is compatible with those of Y_0 and Y_1 along their boundaries. Note that if Y_i is a twisted I–bundle over the Klein bottle for $i = 0,1$, then $K_n(m_n)$ has a Seifert fibration over the base orbifold $S^2(2,2,2,2), \mathbb{R}P^2(2,2)$ or the Klein bottle. In either case $H_1(K_n(m_n))$ is not cyclic, a contradiction. Hence at least one of Y_0 and Y_1, say Y_0, is not a twisted I–bundle over the Klein bottle. So Y_0 has a unique Seifert fibration and Y_1 has at most two Seifert fibrations. Let A be an annulus in $X \cup_{-\frac{1}{n_0}} N(c) = S^1 \times S^1 \times [0,1]$ such that $A \cap T_i$ is a regular fiber of ∂Y_i for some Seifert fibration of Y_i for $i = 1,2$. If the slope of $A \cap T_1$ is distinct from that of $A_1 \cap T_1$, then the annulus twist realizing $(-\frac{1}{n_j})$–surgery on c_0^* changes the slope of $A \cap T_1$ depending on n. Since Y_1 has at most two Seifert fibrations, for at least one of n_1 and n_2, there is no annulus $A \subset X \cup_{-\frac{1}{n_j}} N(c)$ which satisfies $A \cap T_i$ is a regular fiber of ∂Y_i. Thus if $K_n(r_n)$ is Seifert fibered for more than two integers n, the slope of $A \cap T_1$ coincides with that of $A_1 \cap T_1$. This then implies that c_0^* is isotopic to the regular fiber $A_1 \cap T_1$ and $V(K;r)$ is Seifert fibered contrary to assumption. Hence there are at most two integers n such that $X \cup_{-\frac{1}{n}} N(c) \cong S^1 \times S^1 \times [0,1]$. □

(ii) Assume that T is compressible for more than two integers $n \in \mathcal{S}$. Then X is a cable space and the distance between the slope $-\frac{1}{n}$ and that of the fiber slope of X is less than or equal to one [11, Theorem 2.0.1]. Since we have at least three such integers n, the fiber slope of the cable space X coincides with the preferred longitude of c. Hence $X \cup_{-\frac{1}{k}} N(c)$ is a solid torus for any integer k. Let X' be the decomposing piece next to X; we will show that $V(K;r) = X' \cup X$ and X' is Seifert fibered.

First assume for a contradiction that we have yet another decomposing piece $X''(\neq X, X')$ in $V(K;r)$. Again, since $X \cup_{-\frac{1}{n}} N(c)$ is a solid torus (with distinct meridional slopes for each integer n) and gives a Dehn filling of X' for each $n \in \mathcal{S}$, X' cannot be hyperbolic (following the argument used for X). Hence we may assume X' admits a Seifert fibration. If some component of $\partial X' - T$ is compressible in $V(K;r) \cup_{-\frac{1}{n}} N(c)$ for more than two integers $n \in \mathcal{S}$, then [11, Theorem 2.0.1] shows that $X \cup X'$ is a cable space, which is impossible because a cable space is atoroidal. So we may assume that some component T' of $\partial X' - T$ is incompressible in $X' \cup_T (X \cup_{-\frac{1}{n}} N(c))$ for all but at most two integers $n \in \mathcal{S}$. Applying the argument in (i) again implies that $V(K;r)$ is Seifert fibered giving us a contradiction.

So $V(K;r)$ consists of two decomposing pieces X and X', where X is a cable space. It remains to see that X' is a Seifert fibered space. If X' is not Seifert fibered, then since it is a decomposing piece of $V(K;r)$, it is hyperbolic. Note that $K_n(r_n) = X' \cup_T (X \cup_{-\frac{1}{n}} N(c))$, where $X \cup_{-\frac{1}{n}} N(c) = S^1 \times D^2$ for $n \in \mathcal{S}$. Thus $K_n(r_n) = X' \cup_{\gamma_n} (S^1 \times D^2)$ for some slope γ_n on $\partial X'$ which varies depending on n. Then [38, Theorem 1.2] shows that $K_n(r_n)$ is hyperbolic for some integer $n \in \mathcal{S}$, a contradiction. Hence $V(K;r) = X \cup X'$, where X is a cable space (with $\partial X \supset \partial V$) and X' is a Seifert fibered space.

Recalling that the fiber slope of X is a preferred longitude of c, $X \cup_{-\frac{1}{n}} N(c) = S^1 \times D^2$ in which c^* is a cable of a core t of this solid torus. In particular, $X \cup_{-\frac{1}{0}} N(c) = S^1 \times D^2$ in which c is a cable of a core t of this solid torus. Since $K(r) - \mathrm{int}(X \cup_{-\frac{1}{0}} N(c))(= X')$ is Seifert fibered, c is a pseudo-seiferter for (K, r). □

4.2. **Integrality of surgeries.** It has been conjectured that if $K(r)$ is a Seifert fibered manifold, then either r is an integer, or K is a trivial knot, a torus knot or a cable of a torus knot. The next theorem strengthens Corollary 1.4 in [43].

Theorem 4.5. *Let (K, r) be a Seifert surgery with a seiferter or a pseudo-seiferter c. Then either $r \in \mathbb{Z}$ or c is a seiferter and K is either a torus knot (i.e. a 0–bridge braid) or a cable of a torus knot (i.e., a cable of a 0–bridge braid) in $V = S^3 - \mathcal{N}(c)$.*

Proof. Assume that $K(r)$ has a degenerate Seifert fibration. Then the proof of Proposition 2.8 in [12] shows that $K(r)$ is a lens space or a connected sum of two lens spaces. In the former $r \in \mathbb{Z}$ or K is a torus knot [11] and in the latter $r \in \mathbb{Z}$ [22]. So in the following we assume that $K(r)$ has a non-degenerate Seifert fibration $\mathcal{F}$ in which c is a fiber or a cable of some fiber. If c is a fiber in $\mathcal{F}$ (so that c is a seiferter), then [43] ([12, Theorem 2.2]) provides our conclusion.

Now we assume that c is a pseudo-seiferter for (K, r). Let W_0 be a fibered solid tubular neighborhood of a fiber t in $\mathcal{F}$ which contains c in its interior as a cable of t: $W_0 - \mathcal{N}(c)$ is a cable space W in which t is an exceptional fiber of index greater than one. By definition the fiber slope of W coincides with the longitudinal slope λ on $\partial N(c)$. (Hence $W_0(c; -\frac{1}{n}) = W \cup_{-\frac{1}{n}} N(c)$ is a solid torus for infinitely many n.) It should be noted that the Seifert fibration of W (in which the slope of the regular fiber agrees with the cabling slope) does not arise from the Seifert fibration

$\mathcal{F}|_{W_0}$, because c is not a fiber in $\mathcal{F}$. In particular, the slope of the fibration $\mathcal{F}|_{\partial W_0}$ and the cabling slope of W in ∂W_0 are distinct.

Performing λ–surgery on c, S^3 becomes $S^1 \times S^2$ since c is an unknot, and $K(r)$ becomes a reducible manifold with a lens space summand since c is a cabled knot with cabling slope λ. Let us show that $S^1 \times S^2 - K$ is irreducible. Assume for a contradiction that it is reducible. Let S be a reducing 2–sphere. If S is separating, since $S^1 \times S^2$ is prime, S bounds a 3–ball B containing K in $S^1 \times S^2 = V \cup_r N(c)$. If S is non-separating, then take a simple loop α intersecting S just in once. Then the exterior of a regular neighborhood $N(S \cup \alpha)$ of $S \cup \alpha$ is a 3–ball B in $S^1 \times S^2 = V \cup_r N(c)$ containing K. If $\partial B \cap N(c) = \emptyset$, then $K \subset B \subset V$ and $K \cup c$ is a split link, a contradiction. Suppose that $\partial B \cap N(c) \neq \emptyset$. Then we may assume by an isotopy that $\partial B \cap N(c)$ consists of meridian disks of $N(c)$. Take the universal cover $\mathbb{R} \times S^2 \to S^1 \times S^2$ and consider the lift of B in $\mathbb{R} \times S^2$. Using an isotopy which preserves the product structure of $\mathbb{R} \times S^2$, in particular leaves $N(c)$ invariant, we deform B into $(0,1) \times S^2$. This then implies that we have a meridian disk of V which does not intersect K. Thus $K \cup c$ is again a split link, a contradiction. Thus viewing K as a knot in $S^1 \times S^2 = V \cup_\lambda N(c)$ with irreducible exterior, r–surgery on K produces a reducible manifold $V(K;r) \cup_\lambda N(c)$. Since $S^1 \times S^2$ is reducible as well, [23, Theorem 1.2] implies that $r \in \mathbb{Z}$. □

4.3. **Pseudo-seiferters do not have linking number 1.** If (K, r) is a Seifert surgery with a seiferter or a pseudo-seiferter c and r is not an integer, then Theorem 4.5 implies that the linking number between c and K is greater than one. Thus Theorem 2.1 shows that $g(K_n) \to \infty$ as $|n| \to \infty$. Hence to prove Theorem 1.13, we may now focus attention on when r is an integer.

Proposition 4.6. *Assume c is a pseudo-seiferter for the integral Seifert surgery (K, m). Then $|\ell k(K, c)| \neq 1$.*

Remark 4.7. The proof below applies more generally. It shows that for any twist family of surgeries, if the twisting circle c becomes a cable so that its 0–slope becomes the cabling slope, then $|\ell k(K, c)| \neq 1$.

Proof. Express $V(K;m)$ as $X \cup_T X'$ where X is a cable space and X' is a Seifert fibered space. Let λ denote the cabling slope (the fiber slope) of X on both ∂V and T. In $K_n(m_n)$, X fills to a solid torus W_n; let μ_n be the meridional slope of $\partial W_n = T = \partial X'$. Observe that $\Delta(\mu_n, \lambda) = p \geq 2$ for some integer p for all $n \in \mathbb{Z}$ and also that $\mu_n = \mu_0 + np\lambda$. Moreover, we have $K_n(m_n) = X'(\mu_n)$, the μ_n–filling of X'.

Let C be the core of the solid torus W_0 in $K(m) = K_0(m_0) = X'(\mu_0)$. Isotope C so that $C \cap N(K^*) = \emptyset$, where K^* is the surgery dual of K. Then $C \subset K(m) - \mathcal{N}(K^*) = S^3 - \mathcal{N}(K)$. Now we can view C as a knot in S^3 disjoint from K that becomes isotopic to the core of the solid torus W_0 after m–surgery on K. (Furthermore,we may view c as a cable of C in $X'(\mu_0)$.) Then $K_n(m_n) = X'(\mu_n)$ may be presented as Dehn surgery on the link $K \cup C$ where m–surgery is done as before on K and μ_n–surgery is done on C.

Let us now use this to obtain an explicit computation of homology. Let λ_0 be the preferred longitude of C in S^3. Then λ, the cabling slope in $T \subset \partial X$, may be expressed as the slope $p\lambda_0 + q\mu_0$ for coprime integers p, q with $p \geq 2$. Hence $\mu_n = \mu_0 + np(p\lambda_0 + q\mu_0) = np^2\lambda_0 + (npq + 1)\mu_0$. Thus μ_n–surgery is $\frac{npq+1}{np^2}$–surgery in standard coordinates.

To simplify exposition, we apply a "slam-dunk" move [19, p.163]. Let C' be a meridian of C. Then μ_n–surgery on C can be viewed as 0–surgery on C and $\frac{-np^2}{npq+1}$–surgery on C'. Now we may obtain the presentation matrix M_n given below for the homology of $K_n(m_n) = X'(\mu_n)$ from its surgery presentation on the link $K \cup C \cup C'$:

$$M_n = \begin{pmatrix} * & * & 0 \\ * & 0 & 1 \\ 0 & npq+1 & -np^2 \end{pmatrix}$$

Since $K_n(m_n)$ is a rational homology sphere,

$$|H_1(K_n(m_n))| = |\det(M_n)| = |A(-np^2) - B(npq+1)| = |np(Ap+Bq)+B|$$

for some integers A and B. Thus, for all but (at most) one integer n,

$$\begin{aligned} \ell k(K,c)^2 &= |H_1(X(\mu_{n+1}))| - |H_1(X(\mu_n))| \\ &= |(n+1)p(Ap+Bq)+B| - |np(Ap+Bq)+B| \\ &= |p(Ap+Bq)| \end{aligned}$$

Since $p \geq 2$, $\ell k(K,c)^2 \neq 1$ and hence $|\ell k(K,c)| \neq 1$. □

5. Infiniteness of Seifert fibered L-spaces in twist families

As we outlined a proof of Theorem 1.13 in Subsection 1.2, in the final step we show that if $\{(K_n, m_n)\}$ contains infinitely many Seifert L-space surgeries, then $|\ell k(K,c)| > 1$ (Theorem 6.1). Our argument for this step is the subject of Section 6 and requires the preliminary results about sequences of Seifert fibered L-spaces established here.

Recall that if $K(m)$ has a non-degenerate Seifert fibration, then its base surface is S^2 or $\mathbb{RP}^2$ for homological reasons. If $K(m)$ has a degenerate Seifert fibration, then it is a lens space or a connected sum of two lens spaces [12, Proposition 2.8]. When $K(m)$ has a Seifert fibration over $\mathbb{RP}^2$ or a degenerate Seifert fibration over S^2, it is shown in [44] that a twist family $\{(K_n, m_n)\}$ obtained by twisting (K, m) along a seiferter contains infinitely many L-space surgeries.

In this section we consider the case where $K(m)$ has a non-degenerate Seifert fibration over S^2, and give a necessary and sufficient condition for $\{(K_n, m_n)\}$ to contain infinitely many L–space surgeries without assuming $|\ell k(K,c)| = 1$ (Theorem 5.1). This generalizes Theorem 1.4 in [44].

Let us first introduce some notation. A non-degenerate Seifert fibered space over the sphere may be described as $S^2(b, a_1, \dots, a_p)$ where $b \in Z$ and $a_i \in \mathbb{Q}$ with $0 \leq a_i < 1$, (which is commonly written as $M(0; b, a_1, \dots, a_p)$ in [14]). If $a_i = 0$ then it is actually a regular fiber and may be omitted; if $a_i \neq 0$ then it is an exceptional fiber. For a degenerate fiber, we allow $a_i = \infty$.

5.1. **Infinite sequence of Seifert fibered L-spaces $K_n(m_n)$ and its limit.** Following [44] we may express $K_n(m_n)$ as the manifold $Y_n = S^2(b, r_1, \dots, r_{s-1}, r_s(n))$ for some integer b, rational numbers $r_i = \frac{\beta_i}{\alpha_i}$ and $r_s(n) = \frac{n\beta+\beta_s}{n\alpha+\alpha_s}$ for $0 < \beta_i < \alpha_i$ $(i = 1, \dots, s-1)$, $0 \leq \beta_s < \alpha_s$ and $\alpha\beta_s - \beta\alpha_s = 1$.

Denote by $M_c(K,m)$ a 3–manifold obtained by $(m,0)$–surgery on the link $K \cup c$. Note that $M_c(K,m)$ is obtained from the Seifert fibered space $K(m)$ by λ–surgery along a fiber $c \subset K(m)$, where λ is the preferred longitude of $c \subset S^3$. Thus $M_c(K,m)$ admits also a (possibly degenerate) Seifert fibration over S^2. As noted in [44], this manifold arises as the limit of the sequence of $\{K_n(m_n)\}$, i.e. $M_c(K,m) = Y_\infty = S^2(b, r_1, \dots, r_{s-1}, r_s(\infty))$ where $r_s(\infty) = \frac{\beta}{\alpha}$. If $\alpha = 0$, then $r_s(\infty) = \infty$ and $M_c(K,m) = Y_\infty$ is a connected sum of lens spaces, hence it is an L-space [65, 8.1(5)] ([52]).

Theorem 5.1. *Let c be a seiferter for a Seifert surgery (K,m) such that $K(m)$ has a non-degenerate Seifert fibration over S^2. Then the following two conditions are equivalent:*

- *The twist family of surgeries $\{(K_n, m_n)\}$ contains infinitely many L-space surgeries.*
- *$M_c(K,m)$, the result of $(m,0)$–surgery on the link $K \cup c$, is an L-space.*

Proof. Recall that $K(m) = Y_0 = S^2(b, r_1, \dots, r_{s-1}, r_s(0))$, where $0 < r_i < 1$ $(i = 1, \dots, s-1)$ and $0 \le r_s < 1$. Suppose that $s \le 3$. Then the result follows from [44, Theorem 1.4]. The result for $s \ge 4$ follows from Proposition 5.3 below (whose proof will be given in the next subsection). □

Remark 5.2. As mentioned above, when $K(m)$ has a Seifert fibration over $\mathbb{RP}^2$ or a degenerate Seifert fibration over S^2, it is shown in [44] that $\{(K_n, m_n)\}$ has infinitely many L-space surgeries. One may also prove in these situations that $M_c(K,m)$ is necessarily an L-space.

Proposition 5.3. *Suppose that $s \ge 4$. Then Y_n is an L-space for infinitely many integers n if and only if Y_∞ is an L-space.*

Remark 5.4. The "only if" part of Proposition 5.3 does not hold if $s \le 3$. For instance $Y_n = S^2(-1, \frac{1}{2}, \frac{1}{2}, \frac{1}{n})$ is an L-space for any integer n but $Y_\infty = S^2(-1, \frac{1}{2}, \frac{1}{2}) = S^2 \times S^1$ is not an L-space. However, note that this sequence $\{Y_n\}$ is not a twist family of surgered manifolds $\{K_n(m_n)\}$: the linking of K and c must be 0 since $|H_1(Y_n)| = 4$ for all n, but Theorem 1.5 shows that $\{K_n\}$ contains at most three L-space knots.

5.2. **Families of Seifert fibered L-spaces over the 2–sphere.** The goal in this subsection is to prove Proposition 5.3.

When the base surface is S^2, then $K(m)$ is an L-space if and only if it does not carry a horizontal foliation, due to Ozsváth-Szabó and Lisca-Stipsicz [54, 40]. The classification of such Seifert fibered spaces that do not carry a horizontal foliation is due to Eisenbud-Hirsh-Neumann, Jankins-Neumann and Naimi [14, 33, 45]. Theorem 5.5 below brings together these results to classify the Seifert fibered spaces over S^2 that are not L-spaces.

Theorem 5.5 ([54, 40] with [14, 33, 45])**.** *Let Y be a Seifert fibered space over S^2 with $s \ge 3$ exceptional fibers: $Y = S^2(b, r_1, \dots, r_s)$, where b is an integer and $0 < r_1 \le \cdots \le r_s < 1$. Then Y is not an L-space if and only if one of the following conditions holds.*

(1) $-(s-2) \le b \le -2$.

(2) $b = -1$ *and there are relatively prime integers* $0 < a < k$ *with* $a \leq \frac{k}{2}$ *such that* $(r_1, \dots, r_{s-2}, r_{s-1}, r_s) < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$.

(3) $b = -(s-1)$ *and there are relatively prime integers* $0 < a < k$ *with* $a \leq \frac{k}{2}$ *such that* $(1 - r_s, \dots, 1 - r_2, 1 - r_1) < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$.

Fix an integer b, an integer $s \geq 4$, rational numbers $0 < r_1 \leq \dots \leq r_{s-1} < 1$, and the function $r_s(n) = \frac{nu+w}{nt+v}$ for integers t, u, v, w such that $0 \leq w < v$ and $tw - uv = 1$. Also define $r_s(\infty) = \frac{u}{t}$. Consider the family of Seifert fibered spaces $Y_n = S^2(b, r_1, \dots, r_{s-1}, r_s(n))$ for $n \in \mathbb{Z}$ along with $Y_\infty = S^2(b, r_1, \dots, r_{s-1}, r_s(\infty))$.

Before proving Proposition 5.3, we prepare some lemmas. These treat Seifert fibered spaces over S^2 with $s \geq 4$ exceptional fibers. Similar results for $s = 3$ may be found in [44]. We start with Lemmas 5.6 and 5.7 which will be used in the proof of the "only if" part of Proposition 5.3.

If $r_s(\infty)$ is an integer, say z, then we can write $Y_\infty = S^2(b+z, r_1, \dots, r_{s-1}) = S^2(b', r_1, \dots, r_{s-1})$ where $b' = b + z$.

Lemma 5.6. *Suppose that* $s \geq 4$, $r_s(\infty) = z \in \mathbb{Z}$ *and let* $b' = b + z$*. Further assume that* $Y_\infty = S^2(b+z, r_1, \dots, r_{s-1}) = S^2(b', r_1, \dots, r_{s-1})$ *is not an L-space, and hence that* $-(s-2) \leq b' \leq -1$*. Then we have:*

(1) *If* $-(s-3) \leq b' \leq -2$*. then* $S^2(b', r_1, \dots, r_{s-1}, r)$ *cannot be an L-space for* $-1 < r < 1$*.*

(2) *If* $b' = -1$*, then there is* $\varepsilon > 0$ *such that* $S^2(b', r_1, \dots, r_{s-1}, r)$ *is not an L-space for* $-1 < r < \varepsilon$*.*

(3) *If* $b' = -(s-2)$*, then there is* $\varepsilon > 0$ *such that* $S^2(b', r_1, \dots, r_{s-1}, r)$ *is not an L-space for* $-\varepsilon < r < 1$*.*

Proof. First we note that if $r = 0$, then $S^2(b', r_1, \dots, r_{s-1}, 0) = S^2(b', r_1, \dots, r_{s-1})$ is not an L-space by the assumption.

(1) If $0 < r < 1$, then by Theorem 5.5 $S^2(b', r_1, \dots, r_{s-1}, r)$ is not an L-space, because $-(s-2) \leq b' \leq -2$. Assume that $-1 < r < 0$. Then $S^2(b', r_1, \dots, r_{s-1}, r) = S^2(b'-1, r_1, \dots, r_{s-1}, r+1)$ is not an L-space, because $-(s-2) \leq b' - 1 \leq -3$. (If $s = 4$, this case cannot occur.)

(2) Since $S^2(-1, r_1, \dots, r_{s-1})$ is not an L-space, following Theorem 5.5 we have relatively prime integers $0 < a < k$ with $a \leq \frac{k}{2}$ such that $(r_1, \dots, r_{s-2}, r_{s-1}) < (\frac{1}{k}, \dots, \frac{a}{k}, \frac{k-a}{k})$. If $0 < r < \frac{1}{k}$, then Theorem 5.5 shows that $S^2(b', r_1, \dots, r_{s-1}, r)$ is not an L-space. Suppose that $-1 < r < 0$. Then $S^2(b', r_1, \dots, r_{s-1}, r) = S^2(b'-1, r_1, \dots, r_{s-1}, r+1)$. Since $b' - 1 = -2$, Theorem 5.5 shows that $S^2(b', r_1, \dots, r_{s-1}, r) = S^2(b'-1, r_1, \dots, r_{s-1}, r+1)$ is not an L-space. Putting $\varepsilon = \frac{1}{k}$, we have the desired result.

(3) Since $S^2(-(s-2), r_1, \dots, r_{s-1})$ is not an L-space, following Theorem 5.5 we have relatively prime integers $0 < a < k$ with $a \leq \frac{k}{2}$ such that $(1 - r_{s-1}, \dots, 1 - r_2, 1 - r_1) < (\frac{1}{k}, \dots, \frac{a}{k}, \frac{k-a}{k})$. If $0 < r < 1$, then $S^2(b', r_1, \dots, r_{s-1}, r)$ is not an L-space by Theorem 5.5. Suppose that $-\frac{1}{k} < r < 0$. Then $S^2(b', r_1, \dots, r_{s-1}, r) = S^2(b'-1, r_1, \dots, r_{s-1}, r+1) = S^2(-(s-1), r_1, \dots, r_{s-1}, r+1)$. Since $1 - (r+1) < \frac{1}{k}$, by Theorem 5.5 $S^2(b', r_1, \dots, r_{s-1}, r)$ is not an L-space. Now the result follows by putting $\varepsilon = \frac{1}{k}$. $\square$

If $r_s(\infty)$ is not an integer, say $z < r_c < z+1$ for some integer z, then $S^2(b, r_1, \dots, r_{s-1}, r_s(\infty)) = S^2(b+z, r_1, \dots, r_{s-1}, r_s(\infty) - z) = S^2(b', r_1, \dots, r_{s-1}, r_s(\infty) - z)$.

Lemma 5.7. *Suppose that $s \geq 4$, $z < r_s(\infty) < z+1$ for some integer z, and let $b' = b+z$. Further assume that $Y_\infty = S^2(b, r_1, \dots, r_{s-1}, r_s(\infty)) = S^2(b', r_1, \dots, r_{s-1}, r_s(\infty) - z)$ is not an L-space, and hence that $-(s-1) \leq b' \leq -1$. Then we have:*

(1) *If $-(s-2) \leq b' \leq -2$, then $S^2(b', r_1, \dots, r_{s-1}, r)$ is not an L-space if $0 < r < 1$.*

(2) *If $b' = -1$, then there exists $\varepsilon > 0$ such that $S^2(-1, r_1, \dots, r_{s-1}, r)$ is not an L-space for any $0 < r < r_s(\infty) - z + \varepsilon$.*

(3) *If $b' = -(s-1)$, then there exists $\varepsilon > 0$ such that $S^2(-(s-1), r_1, \dots, r_{s-1}, r)$ is an L-space for any $r_s(\infty) - z - \varepsilon < r < 1$.*

Proof. (1) If $0 < r < 1$, then $S^2(b', r_1, \dots, r_{s-1}, r)$ is not an L-space by Theorem 5.5.

(2) Since $S^2(-1, r_1, \dots, r_{s-1}, r_s(\infty) - z)$ is not an L-space, Theorem 5.5 shows that there are relatively prime integers $0 < a < k$ with $a \leq k/2$ such that $(r_1, \dots, r_{s-1}, r_s(\infty) - z)^* < (\frac{1}{k}, \dots, \frac{a}{k}, \frac{k-a}{k})$. Then clearly we can take $\varepsilon > 0$ so that for any $0 < r < r_s(\infty) - z + \varepsilon$, we have $(r_1, \dots, r_{s-1}, r)^* < (\frac{1}{k}, \dots, \frac{a}{k}, \frac{k-a}{k})$. Thus by Theorem 5.5 again $S^2(-1, r_1, \dots, r_{s-1}, r)$ is not an L-space for any $0 < r < r_s(\infty) - z + \varepsilon$.

(3) Since $S^2(-(s-1), r_1, \dots, r_{s-1}, r_s(\infty) - z)$ is not an L-space, by Theorem 5.5 we have relatively prime integers $0 < a < k$ with $a \leq k/2$ such that $(1 - r_1, \dots, 1 - r_{s-1}, 1 - (r_s(\infty) - z))^* < (\frac{1}{k}, \dots, \frac{a}{k}, \frac{k-a}{k})$. Hence there exists $\varepsilon > 0$ such that if $0 < 1 - r < 1 - (r_s(\infty) - z) + \varepsilon < 1$, i.e. $0 < r_s(\infty) - z - \varepsilon < r < 1$, then $(1 - r_1, \dots, 1 - r_{s-1}, 1 - (r_s(\infty) - z))^* < (\frac{1}{k}, \dots, \frac{a}{k}, \frac{k-a}{k})$. Following Theorem 5.5 $S^2(-(s-1), r_1, \dots, r_{s-1}, r)$ is not an L-space for any $0 < r_s(\infty) - z - \varepsilon < r < 1$. □

The following two lemmas will be used in the proof of the "if" part of Proposition 5.3.

Lemma 5.8. *Suppose that $s \geq 4$, $r_s(\infty) = z \in \mathbb{Z}$ and let $b' = b + z$. Further assume that $Y_\infty = S^2(b+z, r_1, \dots, r_{s-1}) = S^2(b', r_1, \dots, r_{s-1})$ is an L-space. Then we have:*

(1) *If $b' \leq -(s-1)$ (resp. $0 \leq b'$), then $S^2(b', r_1, \dots, r_{s-1}, r)$ is an L-space for $-1 < r < 0$ (resp. $0 < r < 1$).*

(2) *If $b' = -1$, then $S^2(b', r_1, \dots, r_{s-1}, r)$ is an L-space for $0 < r < 1$.*

(3) *If $b' = -(s-2)$, then $S^2(b', r_1, \dots, r_{s-1}, r)$ is an L-space for $0 < r < 1$.*

Proof. (1) If $0 \leq b'$ and $0 < r < 1$, then the result follows from Theorem 5.5. If $b' \leq -(s-1)$ and $-1 < r < 0$, then $S^2(b', r_1, \dots, r_{s-1}, r) = S^2(b'-1, r_1, \dots, r_{s-1}, r+1)$. Since $b' - 1 \leq -s$ and $0 < r+1 < 1$, Theorem 5.5 shows that $S^2(b'-1, r_1, \dots, r_{s-1}, r+1)$ is an L-space.

(2) Assume for a contradiction that $S^2(-1, r_1, \dots, r_{s-1}, r)$ is not an L-space for some $0 < r < 1$. By Theorem 5.5 there is a relatively prime integers $0 < a < k$ with $a \leq k/2$ such that $(r_1 \dots, r_{s-1}, r)^* < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$. Then $(r_1 \dots, r_{s-1})^* < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$, which implies that $Y_\infty = S^2(-1, r_1, \dots, r_{s-1})$ is not an L-space contradicting the assumption.

(3) Suppose that $S^2(-(s-2), r_1, \dots, r_{s-1}, r)$ is not an L-space for some $0 < r < 1$. By Theorem 5.5 there is a relatively prime integers $0 < a < k$ with $a \leq k/2$ such that $(1 - r_1 \dots, 1 - r_{s-1}, 1 - r)^* < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$. Then $(1 - r_1 \dots, 1 - r_{s-1})^* < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$, which implies that $Y_\infty = S^2(-(s-2), r_1, \dots, r_{s-1})$ is not an L-space contradicting the assumption. □

Lemma 5.9. *Suppose that $s \geq 4$, $z < r_s(\infty) < z+1$ for some integer z, and let $b' = b+z$. Further assume that $Y_\infty = S^2(b+z, r_1, \dots, r_{s-1}, r_s(\infty)) = S^2(b', r_1, \dots, r_{s-1}, r_s(\infty) - z)$ is an L-space. Then we have:*

1. *If $b' \leq -s$ or $0 \leq b'$, then $S^2(b', r_1, \dots, r_{s-1}, r)$ is an L-space for $0 < r < 1$.*
2. *If $b' = -1$, then $S^2(b', r_1, \dots, r_{s-1}, r)$ is an L-space for $0 < r_s(\infty) - z < r < 1$.*
3. *If $b' = -(s-1)$, then $S^2(b', r_1, \dots, r_{s-1}, r)$ is an L-space for $0 < r < r_s(\infty) - z < 1$.*

Proof. (1) The result follows from Theorem 5.5 immediately.

(2) Assume for a contradiction that $S^2(-1, r_1, \dots, r_{s-1}, r)$ is not an L-space for some $0 < r_s(\infty) - z < r < 1$. Then following Theorem 5.5 we have relatively prime integers $0 < a < k$ with $a \leq k/2$ such that $(r_1 \dots, r_{s-1}, r)^* < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$. Since $r_s(\infty) - z < r$, $(r_1 \dots, r_{s-1}, r_s(\infty) - z)^* < (r_1 \dots, r_{s-1}, r)^* < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$, and hence $Y_\infty = S^2(-1, r_1, \dots, r_{s-1}, r_s(\infty) - z)$ is not an L-space. This contradicts the assumption.

(3) Suppose for a contradiction that $S^2(-(s-1), r_1, \dots, r_{s-1}, r)$ is not an L-space for some $0 < r < r_s(\infty) - z < 1$. By Theorem 5.5 there are relatively prime integers $0 < a < k$ with $a \leq k/2$ such that $(1-r_1 \dots, 1-r_{s-1}, 1-r)^* < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$. Since $r < r_s(\infty) - z$, i.e. $1 - (r_s(\infty) - z) < 1 - r$, $(1-r_1 \dots, 1-r_{s-1}, 1-(r_s(\infty)-z))^* < (1-r_1 \dots, 1-r_{s-1}, 1-r)^* < (\frac{1}{k}, \dots, \frac{1}{k}, \frac{a}{k}, \frac{k-a}{k})$, and hence $Y_\infty = S^2(-(s-1), r_1, \dots, r_{s-1}, r_s(\infty) - z)$ is not an L-space contradicting the assumption. □

Now we are ready to prove Proposition 5.3.

Proof of Proposition 5.3. First we prove the "only if" part of Proposition 5.3. Recall $Y_n = S^2(b, r_1, \dots, r_{s-1}, r_s(n))$ for $n \in \mathbb{Z} \cup \{\infty\}$. Assume Y_∞ is not an L-space. We show that Y_n can be an L-space for only finitely many $n \in \mathbb{Z}$.

If $r_s(\infty) = \infty$, then Y_∞ is a connected sum of lens spaces and hence an L-space [65, 8.1(5)] ([52]), a contradiction. So we assume that $r_s(\infty) = \frac{u}{t}$. Further, for $n \in \mathbb{Z}$, note that $r_s(n) = \frac{nu+w}{nt+v} = r_s(\infty) + \frac{w - r_s(\infty)v}{nt+v}$ so that as a real function $y = r_s(x)$ forms an equilateral hyperbola with $y = r_s(\infty)$ as an asymptotic line.

Assume that $r_s(\infty)$ is an integer z and let $b' = b + z$. Then $Y_\infty = S^2(b, r_1, \dots, r_{s-1}, z) = S^2(b', r_1, \dots, r_{s-1})$, and $Y_n = S^2(b, r_1, \dots, r_{s-1}, r_s(n)) = S^2(b', r_1, \dots, r_{s-1}, r_s(n) - z)$. Since $y = r_s(x) - z$ forms an equilateral hyperbola with $y = 0$ as an asymptotic line, it follows from Lemma 5.6 that there are only finitely many integers n such that $Y_n = S^2(b', r_1, \dots, r_{s-1}, f(n) - q)$ is an L-space.

Finally suppose that $r_s(\infty)$ is not an integer, say $z < r_s(\infty) < z+1$ for some integer z. Again, let $b' = b+z$. By assumption $Y_\infty = S^2(b, r_1, \dots, r_{s-1}, r_s(n)) = S^2(b', r_1, \dots, r_{s-1}, r_s(\infty) - z)$ is not an L-space. Note that $Y_n = S^2(b, r_1, \dots, r_{s-1}, r_s(n)) = S^2(b', r_1, \dots, r_{s-1}, r_s(n) - z)$ and $y = r_s(x) - z$ forms an equilateral hyperbola with $y = r_s(\infty) - z$ as an asymptotic line. Then Lemma 5.7 shows that there are only finitely many integers n such that $Y_n = S^2(b', r_1, \dots, r_{s-1}, r_s(n) - z)$ is an L-space.

Next we prove the "if" part of Proposition 5.3

Assume that $r_s(\infty) = \infty$, i.e. $t = 0, u = -1, v = 1$. Then $r_s(n) = \frac{nu+w}{nt+v} = -n + w$. Hence $Y_n = S^2(b, r_1, \dots, r_{s-1}, r_s(n)) = S^2(b, r_1, \dots, r_{s-1}, -n+w) = S^2(b+w-n, r_1, \dots, r_{s-1})$. Following

Theorem 5.5 Y_n is an L-space if $b+w-n \le -(s-1)$ or $0 \le b+w-n$. Thus Y_n is an L-space for infinitely many n.

Suppose that $r_s(\infty)$ is an integer z and let $b' = b+z$. As above, $Y_n = S^2(b, r_1, \dots, r_{s-1}, r_s(n)) = S^2(b', r_1, \dots, r_{s-1}, r_s(n)-z)$ and $y = r_s(x) - z$ forms an equilateral hyperbola with $y = 0$ as an asymptotic line. Hence Lemma 5.8 shows that Y_n is an L-space for infinitely many integers n.

Finally suppose that $r_s(\infty)$ is not an integer; $z < r_s(\infty) < z+1$ for some integer z. $Y_n = S^2(b, r_1, \dots, r_{s-1}, r_s(n)) = S^2(b', r_1, \dots, r_{s-1}, r_s(n)-z)$. Since $y = r_s(x) - z$ forms an equilateral hyperbola with $y = r_s(\infty) - z$ as an asymptotic line, it follows from Lemma 5.9 that Y_n is an L-space for infinitely many integers n.

This completes a proof of Proposition 5.3. $\square$(Proposition 5.3)

6. L-space knots in twist families with linking number one

Our goal in this section is to prove:

Theorem 6.1. *Let $\{(K_n, m_n)\}$ be a twist family of surgeries, i.e. (K_n, m_n) is a surgery obtained from (K, m) by twisting along a trivial knot c. Suppose that $\{(K_n, m_n)\}$ contains infinitely many Seifert L-space surgeries. Then $|\ell k(K, c)| > 1$.*

Proof of Theorem 6.1. Since $\{(K_n, m_n)\}$ contains more than three L-space surgeries, $\ell k(K, c) \neq 0$ (Theorem 1.5). So it remains to show that $|\ell k(K, c)| \neq 1$. Since $\{(K_n, m_n)\}$ contains more than nine Seifert surgeries, we may reparametrize so that (K, m) is a Seifert surgery and then by Theorem 4.2 c is a seiferter or a pseudo-seiferter for (K, m). If c is a pseudo-seiferter, Proposition 4.6 shows $|\ell k(K, c)| \neq 1$. So in the following we assume that c is a seiferter for (K, m). Then by the inheritance property (K_n, m_n) is a Seifert surgery for all n. Recall that $m = m_0$ and $m_n = m_0 + n\ell k(K, c)^2 \ge m_0 + n$ since $\ell k(K, c) \neq 0$. Thus we may assume there is a Seifert surgery (K_n, m_n) with $m_n \neq 0$ and reparametrize so that this is (K, m).

In accordance with the possible types of Seifert fibrations of $K(m)$ for $m \in \mathbb{Z}$, $m \neq 0$, we divide into four cases:

Case I. $K(m)$ is non-degenerate Seifert fibered space over S^2.
Case II. $K(m)$ is non-degenerate Seifert fibered space over $\mathbb{RP}^2$.
Case III. $K(m)$ is a connected sum of two lens spaces.
Case IV. $K(m)$ is a lens space with a degenerate Seifert fibration.

Recall that $\{(K_n, m_n)\}$ contains infinitely many L-space surgeries. For each of above cases we show that either $|\ell k(K, c)| \neq 1$ or a reparametrization of our twist family puts us in a previous case. Recall also that we assume c neither bounds a disk disjoint from K nor is a meridian of K.

6.1. **Case I: $K(m)$ is a non-degenerate Seifert fibered space over S^2.** Since $K_n(m_n)$ is an L-space for infinitely many integers n, Theorem 5.1 shows that $M_c(K, m)$ is an L-space.

Lemma 6.2. *Let c be a seiferter for (K, m) with $|\ell k(K, c)| = 1$. If $M_c(K, m)$ is an L-space, then $M_c(K, m)$ is either S^3 or the Poincaré homology 3–sphere $\Sigma(2, 3, 5)$.*

Proof. Since $M_c(K,m)$ is the result of $(m,0)$–surgery on the link $K \cup c$, $H_1(M_c(K,m)) = \langle\, \mu_c,\ \mu_K \mid w\mu_c + m\mu_K = 0,\ w\mu_K = 0 \,\rangle$, where μ_c is a meridian of c, μ_K is that of K, and $\omega = \ell k(K,c)$. Since $|\omega| = 1$, $M_c(K,m)$ is an integral homology 3–sphere. In particular, $M_c(K,m)$ is not a degenerate Seifert fibered space, i.e. a connected sum of lens spaces. Thus $M_c(K,m)$ is Seifert fibered homology 3–sphere. Since $M_c(K,m)$ is an L-space as well, it is either S^3 or the Poincaré homology 3–sphere $\Sigma(2,3,5)$. See [6, Proposition 2.2] and [8, Theorem 1.1]. □

In the following we show that $M_c(K,m)$ is neither S^3 nor $\Sigma(2,3,5)$. Since we assume c is not a meridian of K, Lemma 6.3 below exclude the possibility of $M_c(K,m) = S^3$.

Lemma 6.3. *Assume $|\ell k(K,c)| = 1$ and $M_c(K,m) = S^3$. Then K is a torus knot and c is a meridian of K.*

Proof. Recall that c is a seiferter for (K,m) such that $|\ell k(K,c)| = 1$. Since $|\ell k(K,c)| = 1$, $(-m)$–twist along c changes (K,m) to $(K_{-m},0)$. Let us re-write $K = K_{-m}$. Thus c is a seiferter for $(K,0)$. Then we may view K as a knot in $V = S^3 - \mathcal{N}(c)$ such that $V(K;0)$ is a Seifert fibered space. Performing $(-1/n)$–surgery along c takes K with its slope 0 to a knot K_n with a Seifert surgery slope n; the twist family is now expressed as $\{(K_n,n)\}$. Observe that $V(K;0) \cong V(K_n;n)$ for all n.

Let λ be the preferred longitude of $c \subset S^3$, i.e. the 0–slope and a meridian of the solid torus V. Let c_n be the surgery dual to the c with meridian μ_n, the $(-1/n)$–surgery slope of c in ∂V. These curves μ_n are each longitudes of V and satisfy $[\mu_n] = [\mu_0] - n[\lambda] \in H_1(\partial V)$.

Recall that $M_c(K,0)$ is the manifold obtained from the Seifert fibered space $K(0)$ by 0–surgery on c. Equivalently, since c is an unknot, 0–surgery on c is $S^1 \times S^2$ and $M_c(K,0)$ is the result of 0–surgery on the image of K in $S^1 \times S^2$.

The above reparametrization offers the convenience that the surgery curve γ_n of slope n for $K_n \subset V \subset S^3$ in $\partial N(K_n)$ is homologous in $V - \mathcal{N}(K_n)$ to μ_n in ∂V. Furthermore μ_0 is null-homologous in $V(K,0)$.

Let c^* be the surgery dual to c (with respect to λ–surgery). Since $M_c(K,0) \cong S^3$, c^* is a knot in S^3 with Seifert fibered exterior $V(K;0)$ and meridian λ. Thus c^* is a torus knot $T_{p,q}$, $V(K;0) \cong S^3 - \mathcal{N}(T_{p,q})$, and μ_0 is the boundary of its Seifert surface. Thus $(\lambda, -\mu_0)$ is a meridian-longitude basis for c^*. Therefore $K_n(n) = c^*(n) = T_{p,q}(n)$ for all integers n. In particular, for each integer n with $|n| > 30(p^2-1)(q^2-1)/67$, $K_n(n) = T_{p,q}(n)$. Then by Ni-Zhang [50, Theorem 1.3], we can conclude that $K_n = T_{p,q}$ for infinitely many integers n. Consequently, $K \cup c$ cannot be hyperbolic by Thurston's Hyperbolic Dehn Surgery Theorem [67, 68, 2, 57, 7]. By taking $n = pq$, $K_{pq}(pq) = T_{p,q}(pq)$, which is a connected sum of two (nontrivial) lens spaces. Note that $K_{pq}(pq) - \mathcal{N}(c) = M_c(K,0) - \mathcal{N}(c^*) = V(K_{pq}, pq)$ and c is a degenerate fiber in $K_{pq}(pq)$. It follows from [12, Corollary 3.21] that we have one of the following: (i) K_{pq} is a torus knot and c is a meridian of K_{pq}, or (ii) $S^3 - K_{pq} \cup c \cong S^3 - K \cup c$ is hyperbolic. Since $K \cup c$ is not a hyperbolic link as we mentioned above, the latter case cannot occur, and hence K_{pq} is a torus knot and c is its meridian. This means also that $K(= K_{pq})$ is a torus knot and c is its meridian. □

The general result below shows that $M_c(K,m)$ is not $\Sigma(2,3,5)$.

Lemma 6.4. *Let $K \cup c \subset S^3$ be a two-component link where c is the unknot. The Poincaré homology sphere $\Sigma(2,3,5)$ cannot be obtained by $(m,0)$–surgery on $K \cup c$ for any integer m.*

Proof. Assume to the contrary that there is such a link $K \cup c$ for which $(m,0)$–surgery produces $\Sigma(2,3,5)$. Since $\Sigma(2,3,5)$ is a homology sphere, as shown in the proof of Lemma 6.2, $|\ell k(K,c)| = 1$. Viewing $(m,0)$–surgery on $K \cup c$ as 0–surgery on c producing $S^1 \times S^2$ followed by m–surgery on the image of K allows us to construct a smooth homology 4–ball W whose boundary is $\Sigma(2,3,5)$. The circle c with its 0–surgery may be regarded as a "dotted circle" defining a 4–dimensional 1–handle attached to B^4. Then the integral surgery on K defines the subsequent 4–dimensional 2–handle attachment that, since the linking number is 1, kills the homology introduced by the 1–handle. However, since $\Sigma(2,3,5)$ has non-trivial Rokhlin invariant, there cannot be a smooth homology 4–ball W with $\partial W = \Sigma(2,3,5)$; see [60, 61]. □

Combining Lemmas 6.2, 6.3 and 6.4 we see that $|\ell k(K,c)| \neq 1$. This completes a proof of Case I.

6.2. **Case II: $K(m)$ is a non-degenerate Seifert fibered space over $\mathbb{RP}^2$.** Recall that since $K(m) = K_0(m_0)$ is a Seifert fibered space over $\mathbb{RP}^2$, $K(m)$ contains a Klein bottle and hence $m = 4k$ for some integer k [66, Lemma 2.4]. Since c is a fiber of this fibration, $K_n(m_n)$ is a Seifert fibered space over $\mathbb{RP}^2$ as well for each n. Hence there are integers k_n such that $m_n = 4k_n$. Yet since $m_n = m + n\ell k(K,c)^2$, this implies $\ell k(K,c)$ is even, in particular, $|\ell k(K,c)| \neq 1$.

6.3. **Case III: $K(m)$ is a connected sum of two lens spaces.** Recall that c is a fiber in a degenerate Seifert fibration $\mathcal{F}$ of a connected sum of two lens spaces $K(m)$.

We divide into two cases depending on c is a non-degenerate fiber or a degenerate fiber in $\mathcal{F}$.

Case IIIa. c is a non-degenerate fiber in $\mathcal{F}$.

Since c is a non-degenerate fiber, c is an exceptional fiber [12, Corollary 3.21], and $V(K,m)$ has a (nontrivial) lens space summand. The reducibility of $V(K,m)$ implies that K is (p,q)–cabled in V, where $p \geq 2$ [62]. Then $\ell k(K,c) = pk$ for some integer $k \geq 0$; hence $|\ell k(K,c)| \neq 1$.

Case IIIb. c is a degenerate fiber in $\mathcal{F}$.

Let us suppose for a contradiction that $|\ell k(K,c)| = 1$. As in [44, Section 2]([12]), $K_n(m_n)$ has a Seifert invariant $S^2(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{n\beta+1}{n})$, and $M_c(K,m) = S^2(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \beta)$. Note that $M_c(K,m)$ is a lens space. Since $|\ell k(K,c)| = 1$, it follows from the argument in the proof of Lemma 6.2 that $M_c(K,m)$ is an integral homology 3–sphere. But since $M_c(K,m)$ is a lens space, it is actually S^3. Now Lemma 6.3 implies that K is a torus knot and c is a meridian of K. This contradicts the assumption.

6.4. **Case IV: $K(m)$ is a lens space with a degenerate Seifert fibration.** Recall that c is a fiber in a degenerate Seifert fibration $\mathcal{F}$ of a lens space $K(m)$. There are at most two degenerate fibers in $K(m)$ [12, Proposition 2.8].

In the case where there are exactly two degenerate fibers, $(K,m) = (O,0)$ where O is the unknot and the exterior of these two degenerate fibers is $S^1 \times S^1 \times [0,1]$. If c is a non-degenerate fiber in $\mathcal{F}$, then $K_n(m_n)$ has $S^1 \times S^2$ as a connected summand for all integers n, and thus $(K_n, m_n) = (O,0)$

for all n [16, Theorem 8.1]; this contradicts [12, Theorem 5.1]. If c is a degenerate fiber in $\mathcal{F}$, then by changing the Seifert fibration of $S^1 \times S^1 \times [0,1]$ we may assume that $K(m) = O(0)$ has a non-degenerate Seifert fibration which contains c as a fiber. Now we are in the situation of Case I.

Let us assume that $K(m)$ has exactly one degenerate fiber t. We have two cases: $K(m) - \mathcal{N}(t)$ either is a Seifert fibered solid torus or has a non-degenerate Seifert fibration over the Möbius band.

Suppose first that $K(m) - \mathcal{N}(t)$ is a fibered solid torus. If c is isotopic to the core of the solid torus $K(m) - \mathcal{N}(t)$, then $V(K; m) = K(m) - \mathcal{N}(c)$ is a solid torus. By [17] K is a 0– or 1–bridge braid in $V(= S^3 - \mathcal{N}(c))$. If $|\ell k(K,c)| = 1$, then c bounds a disk which intersects K just once, contrary to assumptions. Hence $|\ell k(K,c)| \neq 1$. If c is not isotopic to the core, then c is a regular fiber in the exceptionally fibered solid torus $K(m) - \mathcal{N}(t)$ whose core is an exceptional fiber of index $p \geq 2$. Since c is a regular fiber in this degenerate Seifert fibration, it is a meridian of t and hence bounds a disk in the lens space $K(m)$. Therefore we may assume that $K_n(m_n)$ is a connected sum of two lens spaces for some integer n. If not, then $(-\frac{1}{n})$–surgery of $K(m) - \mathcal{N}(t)$ on c yields a fibered solid torus whose core is an exceptional fiber of index p. This implies $H_1(K_n(m_n)) \cong \mathbb{Z}_p$ for all n implying that $\ell k(K,c) = 0$ contradicting Theorem 1.5. Thus, after reparametrizing our twist family, we are in the situation of Case III.

Now suppose that $K(m) - \mathcal{N}(t)$ has a non-degenerate Seifert fibration over the Möbius band. If c is a non-degenerate fiber, then $K_n(m_n)$ has $S^1 \times S^2$ as a connected summand for all n. Thus $(K_n, m_n) = (O, 0)$ for all n [16, Theorem 8.1], which is impossible [12, Theorem 5.1]. So c is a degenerate fiber t. Then after 1–twist, $K_1(m_1)$ is a Seifert fibered space over $\mathbb{RP}^2$ and c is a fiber in a non-degenerate Seifert fibration of $K_1(m_1)$. Thus a reparametrization puts us in the situation of Case II. This completes the proof of Case IV.

Thus the proof of Theorem 6.1 is completed. □(Theorem 6.1)

7. Positive braids and seiferters for torus knots

7.1. **Genera of positive braid closures.**

Proposition 7.1. *There are only finitely many knots of each genus that are closures of positive braids.*

Proof. Observe that there are exactly $(n-1)^\ell$ positive braids in B_n with word length ℓ. Also, if a positive braid $\beta \in B_n$ has word length ℓ, then the oriented closed braid $\widehat{\beta}$ is a fibered link which bounds a Seifert surface (a fiber surface) with Euler characteristic $\chi(\widehat{\beta}) = n - \ell$ [64, Theorem 2]. Furthermore, if $\ell < 2(n-1)$ then either $\widehat{\beta}$ is a split link of at least two components or there is a positive braid $\beta' \in B_{n-1}$ such that $\widehat{\beta} = \widehat{\beta}'$. This is because the bound $\ell < 2(n-1)$ implies that some generator of B_n either does not appear in β at all or only appears once. In the former, the closure necessarily is a split link; in the latter, β admits a Markov type of destabilization to β'. See figure 7.1 for an illustration of these two cases.

Now assume K is a knot that is the closure of a positive braid. Let n be the smallest index such that $K = \widehat{\beta}$ for a positive braid $\beta \in B_n$. Since a fibered knot has a unique Seifert surface [13, Lemma 5.1] ([69]), $g(K) = (1 - \chi(\widehat{\beta}))/2 = (1 - n + \ell)/2$, and the word length of β is

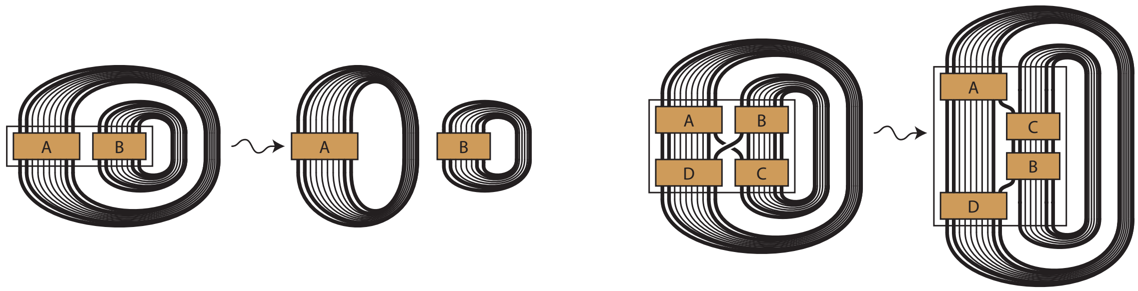

FIGURE 7.1. If a generator of B_n does not appear in the braid word β, the link $\widehat{\beta}$ is split. If a generator appears just once in the positive braid word β, there is positive braid word β' of smaller braid index such that $\widehat{\beta'} = \widehat{\beta}$.

$\ell = 2g(K) + n - 1 \geq 2(n-1)$. Hence $2g(K) + 1 \geq n$. Thus the positive braid index of a knot is bounded above by its genus. Therefore for each braid index there are only finitely many positive braids whose closure is a knot of a given genus. □

7.2. **Torus knots and the seiferter c_+.** Recall that torus knots $T_{p,q}$ are fundamental examples of L-space knots. In the standardly embedded torus T with preferred oriented meridian m and longitude l, the torus knot $T_{p,q}$ is the unoriented curve in T homologous to $\pm(q[l] + p[m]) \in H_1(T)$ when given an orientation. We may then choose that $q > 0$ so that $T_{p,q}$ is a positive braid and a positive L-space knot whenever $p > 0$ and a negative braid and a negative L-space knot whenever $p < 0$. We say $T_{p,q}$ is a positive torus knot in the former situation and a negative torus knot in the latter. Since $g(T_{p,q}) = (|p| - 1)(q - 1)/2$, for any given integer N, there are only finitely many $T_{p,q}$ with $g(T_{p,q}) \leq N$.

Depicted on the left and right side of Figure 7.2 are unknots c_+ and c_- disjoint from a once-punctured T, though the right side shows $-l$. Given the torus knot $T_{p,q}$ in the punctured torus T, we define $c_\pm$ to be the corresponding knot in the complement of $T_{p,q}$. It follows from [12] (where they are called $c^{\pm}_{p,q}$) that these are seiferters for the torus knots $T_{p,q}$ with the pq surgery. The central two images of Figure 7.2 shows that $T_{p,q} \cup c_+$ is the mirror of $T_{-p,q} \cup c_-$; the mirroring is through a vertical plane containing the curve m. Hence, by mirroring as needed, we may restrict attention to the seiferter c_+.

Define $T_{p,q,n}$ to be the result of an n–twist of the torus knot $T_{p,q}$ along the seiferter c_+. Note that if $p = 0$ or $q = 0$ then $T_{p,q,n}$ is an unknot for all n. Furthermore, we may choose $q > 0$ since the knot $T_{p,q,n}$ is unoriented.

Theorem 1.7 of [44] and its proof show that

- if $p > 0$ then $T_{p,q,n}$ is an L-space knot for all integers n, and
- if $p < 0$ then $T_{p,q,n}$ is an L-space knot for any integer $n \leq 1$.

Furthermore, in either case, $|\ell k(T_{p,q,n}, c_+)| = |p+q|$ and the algebraic linking equals the geometric linking, i.e. $T_{p,q,n}$ intersects a disk bounded by c_+ in the same direction. See Figure 7.3. Thus Theorem 2.1 shows that $g(T_{p,q,n}) \to \infty$ as $|n| \to \infty$ for any p, q with $|p + q| > 1$. (If $|p + q| = 1$, then c_+ is a meridian of $T_{p,q,n}$. If $|p+q| = 0$, then $-p = q = 1$ and $T_{-1,1,n} \cup c_+$ is the unlink.) Hence Conjecture 1.2 is satisfied for each twist family of knots $T_{p,q,n}$ individually.

However, by showing the L-space knots among all the knots $T_{p,q,n}$ are positive or negative braids, we may conclude that Conjecture 1.2 is satisfied for the twist families of knots $T_{p,q,n}$ collectively.

Lemma 7.2.

- *If $p > 0$ then $T_{p,q,n}$ is a positive or negative braid for all integers n.*
- *If $p < 0$ then $T_{p,q,n}$ is a positive or negative braid for any integer $n \leq 2$.*

Corollary 7.3. *Conjecture 1.2 is satisfied for the collection of knots $T_{p,q,n}$ with either $p > 0$, $q > 0$, and all n or $p < 0$, $q > 0$ and $n \leq 2$.*

Proof. This follows immediately from Proposition 7.1 and Lemma 7.2. □

Proof of Lemma 7.2. We represent the torus knot $T_{p,q}$ in the once punctured torus T by one of two train tracks in T depending on whether $p > 0$ or $p < 0$ (and requiring $q > 0$). The knots carried by these train tracks after $(-1/n)$–surgery on c_+ are our knots $T_{p,q,n}$. By a sequence of isotopies of T, c_+, and the train tracks along with splittings of the train tracks we will arrange the train tracks into positions where it is apparent that they carry positive or negative braids after $(-1/n)$–surgery on c_+ for particular values of n.

For $p > 0$, Figure 7.4 shows that the knot $T_{p,q,n}$ is actually a positive braid if $n \geq 0$ and a negative braid if $n \leq -1$.

For $p < 0$, Figure 7.5 shows that the knot $T_{p,q,n}$ is a negative braid if $n \leq 0$. Continuing from this, Figure 7.6 indicates how to further isotope $T_{p,q,n}$ (with $p < 0$) into a positive or negative braid for $n = 1$ or $n = 2$.

First assume $n = 2$. If $2q \geq |p|$ (as on the right side of Figure 7.6), then the knot may be isotoped to a negative braid. If $|p| \geq 2q$ (as on the left side of Figure 7.6), then the knot may be isotoped to a positive braid appear.

Now assume $n = 1$. Then we may discard the twisting circle in Figure 7.6. If $2q \geq |p|$ (as on the right side of Figure 7.6), then the knot may be isotoped to a negative braid. If $|p| \geq 2q$ (as on the left side of Figure 7.6), then the knot may be isotoped into a configuration similar to the initial configuration at the top of Figure 7.6, but with smaller braid index and mirrored (say, mirrored across a horizontal line below the diagram). This argument can now be repeated until a positive or negative braid is achieved. □

Question 7.4. *Which knots $T_{p,q,n}$ with $p < 0$, $q > 0$, and $n \geq 2$ are L-space knots?*

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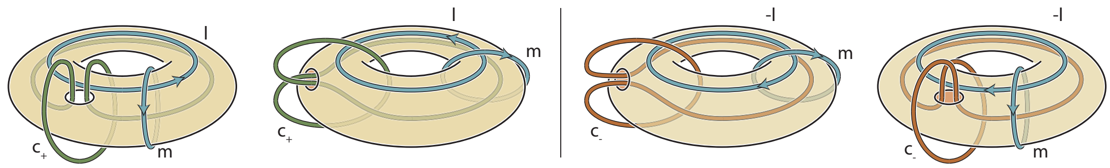

FIGURE 7.2. The seiferter c_+ for a (p, q)–torus knot is mirror equivalent to the seiferter c_- for a $(-p, q)$–torus knot.

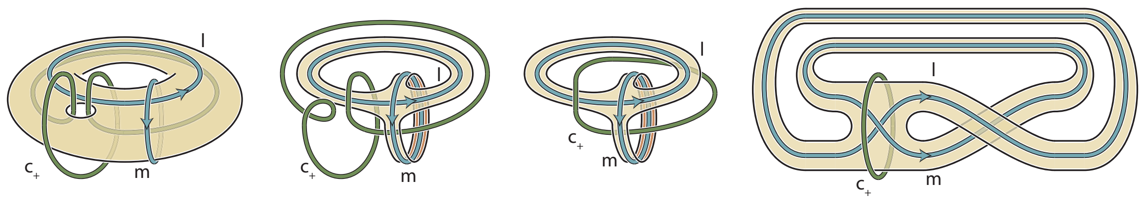

FIGURE 7.3. The standardly embedded once-punctured torus, its preferred meridian-longitude basis, and the seiferter c_+ for the torus knots carried by this torus are isotoped into a convenient configuration.

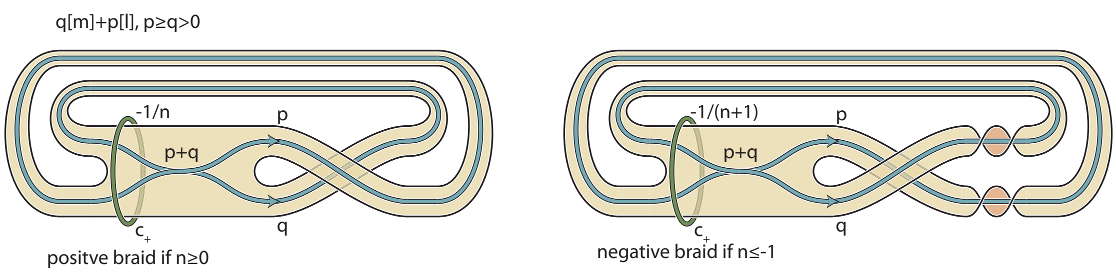

FIGURE 7.4. Beginning with a positive torus knot, $(-1/n)$–surgery on the seiferter c_+ produces a closed positive braid if $n \geq 0$ and a closed negative braid if $n \leq -1$.

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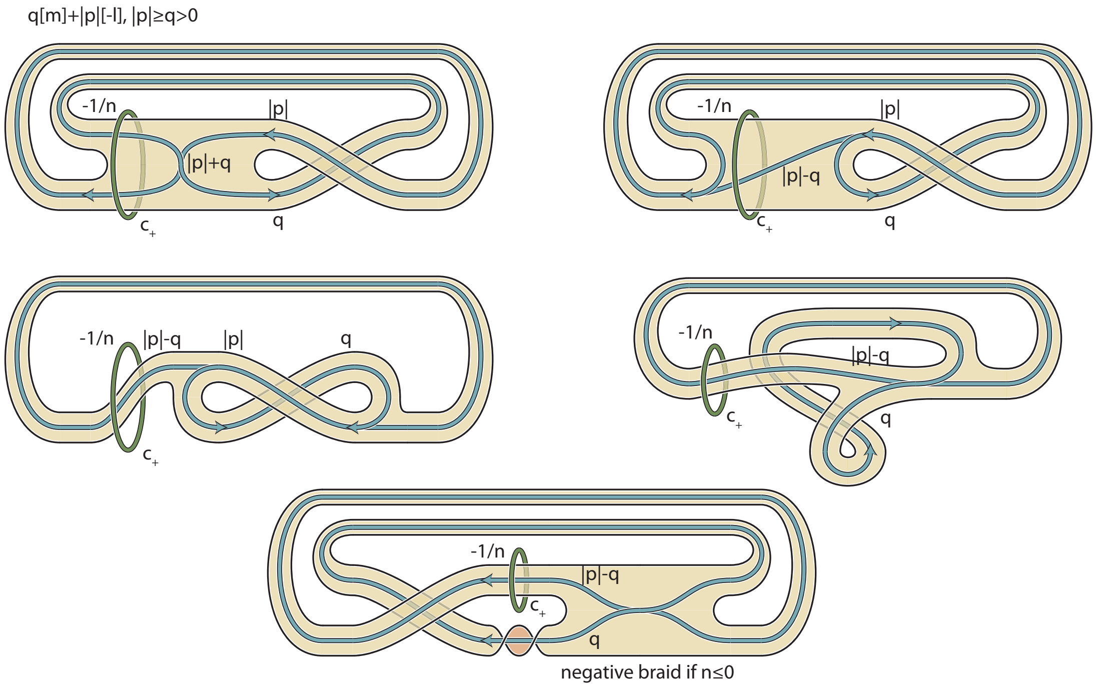

FIGURE 7.5. Beginning with a negative torus knot, $(-1/n)$–surgery on the seifer- ter c_+ produces a closed negative braid if $n \leq 0$.

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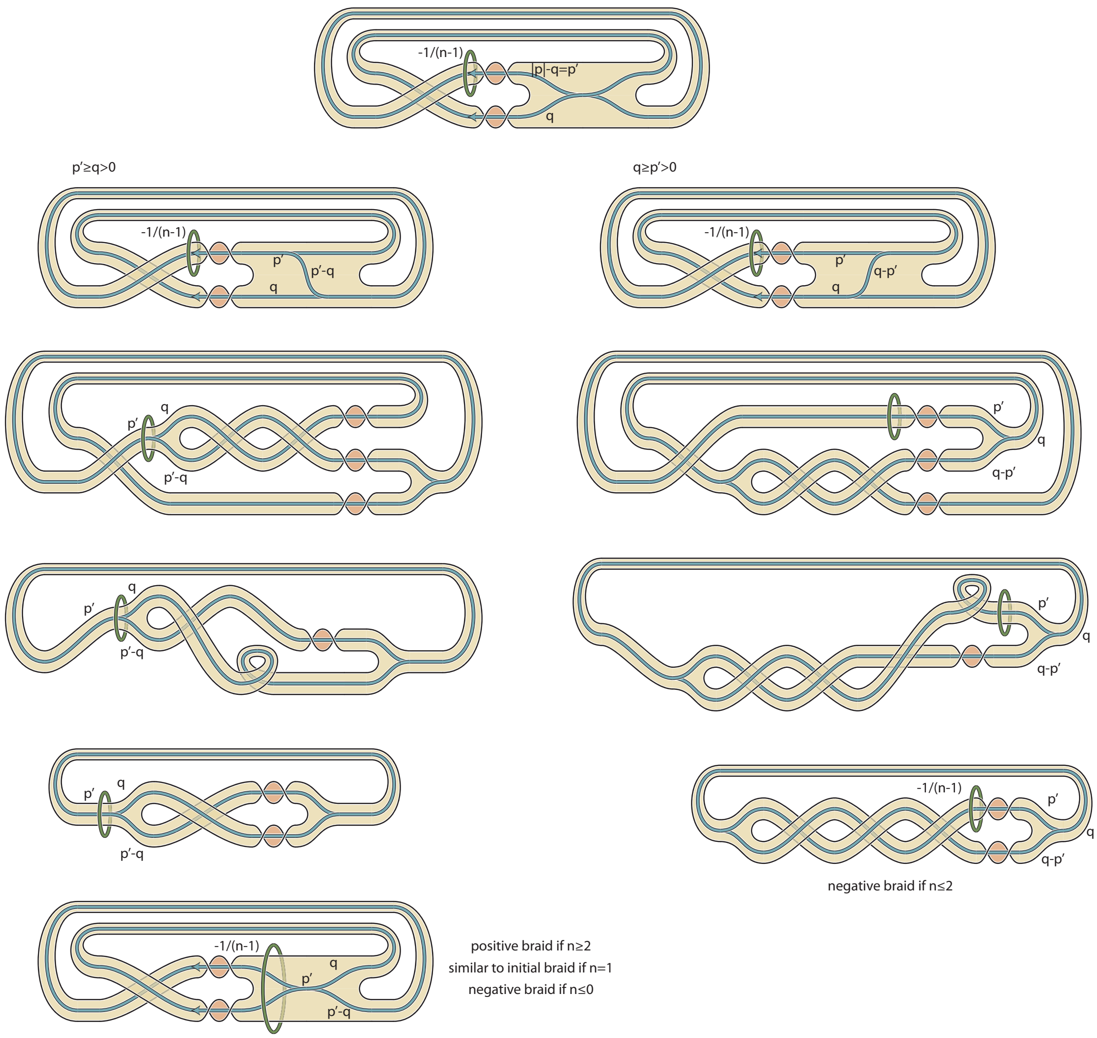

Figure 7.6. Continuing from the end of Figure 7.5, two possibilities are examined. Down the left, when $p' = |p| - q \geq q$: either $n \neq 1$ and the knot can be isotoped to a positive or negative braid; or $n = 1$ and the knot can be rearranged into a mirrored form of the initial position but with smaller braid index. Down the right, when $q \geq p'$, the knot can be isotoped into a negative braid if $n \geq 2$.

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