BAND SUM OPERATIONS YIELDING TRIVIAL KNOTS

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ABSTRACT. Let $L$ be a 2-component link in the 3-sphere $S^3$ consisting of a knot $K$ and its meridian $c$. Let $b : [0, 1] \times [0, 1] \to S^3$ be an embedding such that $b([0, 1] \times [0, 1]) \cap K = b([0, 1] \times \{0\})$ and $b([0, 1] \times [0, 1]) \cap c = b([0, 1] \times \{1\})$. Then we obtain a knot $L_b$ by replacing $b([0, 1] \times \{0\})$ in $L$ with $b([0, 1] \times [0, 1])$. We call $L_b$ a band sum of $L$ with the band $b$. If $K$ is a trivial knot, i.e. $L$ is a Hopf link, then Thompson has proved that only obvious band can create a trivial knot $L_b$. In the present paper we will show that $K$ is a nontrivial knot and $L_b$ is a trivial knot for some band $b$ if and only if $K$ has unknotting number one. As a particular case, if $K$ is a torus knot, we determine the band $b$ with $L_b$ a trivial knot.

1. Introduction

Let $L = K_1 \cup K_2$ be a 2-component link in the 3-sphere $S^3$. Let $b : [0, 1] \times [0, 1] \to S^3$ be an embedding such that $b([0, 1] \times [0, 1]) \cap K_1 = b([0, 1] \times \{0\})$ and $b([0, 1] \times [0, 1]) \cap K_2 = b([0, 1] \times \{1\})$. Then we obtain a knot $L_b$ by replacing $b([0, 1] \times \{0\})$ in $L$ with $b([0, 1] \times [0, 1])$, see Figure 1. We call $L_b$ a band sum of $L$ with the band $b$. In the following, for simplicity, we use the same symbol $b$ to denote the image $b([0, 1] \times [0, 1])$.

![Figure 1. Band sum operation](image)

It follows from [6] and [8] that we completely understand when we can obtain a trivial knot from a split link, i.e. there is a 2-sphere (called splitting sphere) in $S^3$ which separates $K_1$ and $K_2$, by band sum operation.

**THEOREM (1.1) (Scharlemann [6], Thompson [8]).** Let $L$ be a 2-component split link. If a band sum $L_b$ is a trivial knot in $S^3$, then both $K_1$ and $K_2$ are unknotted and the band $b$ is trivial, i.e. there is a splitting 2-sphere $S$ so that $b \cap S$ consists of a single arc.

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Now we start with a 2-component link obtained from a split link by a single crossing change in different components. To make precise we say that a 2-component link \( L = K_1 \cup K_2 \) is a Hopf sum of \( K_1 \) and \( K_2 \) and denote by \( K_1 \cup_H K_2 \) if \( L \) is obtained from \( K_1, K_2 \) and the Hopf link by connected sum operation in a suitable way, see Figure 2. This operation depends on the orientations of \( K_1, K_2 \) and the Hopf link, but in the following such an ambiguity is irrelevant.

![Figure 2. Hopf sum \( K_1 \cup_H K_2 \)](image)

In the case where both \( K_1 \) and \( K_2 \) are unknotted, i.e. \( L \) is a Hopf link \( H \), Thompson [9], Corollary 3, has proved the following. See [3] for an alternate proof and a generalization where \( L \) is a \((2, 2p)\)-torus link.

**Theorem (1.2) (Thompson).** Let \( L \) be a Hopf link. Then \( L_b \) is a trivial knot if and only if \( L \cup b(\{\frac{1}{2}\} \times [0, 1]) \) has a planar projection with exactly two crossings and \( b \) is untwisted or half-twisted.

Furthermore, following Eudave-Muñoz [1], Corollary 2, we have:

**Theorem (1.3) (Eudave-Muñoz).** Let \( L \) be a Hopf sum \( K_1 \cup_H K_2 \) of \( K_1 \) and \( K_2 \).

1. If both \( K_1 \) and \( K_2 \) are knotted, then \( L_b \) is knotted for any band \( b \).
2. If \( K_1 \) is a composite knot and \( K_2 \) is a trivial knot, then \( L_b \) is knotted for any band \( b \).

![Figure 3. Band sum of a Hopf sum](image)

In the present note we will consider the remaining case: \( K_1 \) is a (nontrivial) prime knot and \( K_2 \) is a trivial knot. Then \( L \) consists of the prime knot \( K_1 = K \) and its meridian \( K_2 = c \), see Figure 3.
**Theorem (1.4).** Let $L$ be a 2-component link $K \cup c$ consisting of a prime knot $K$ and its meridian $c$. Then a band sum $L_b$ is a trivial knot for some band $b$ if and only if $K$ has unknotting number one.

Theorem (1.4), together with a result of Kobayashi [4] ([7]), we have the following result which is motivated by a study of Seifert surgery on knots [2].

**Theorem (1.5).** Let $L$ be a 2-component link $T_{p,q} \cup c$ consisting of a $(p,q)$-torus knot $T_{p,q}$ ($|p| > q \geq 2$) and its meridian $c$. If a band sum $L_b$ is a trivial knot, then $(p,q) = (\pm 3, 2)$ and the band $b$ is given by Figure 4 up to isotopy in $S^3$; the isotopy is not necessarily leaving the link $L$ invariant.

![Figure 4. Trivializing band for $T_{p,q} \cup c$](image)

**2. Proof of Theorem (1.4)**

*Proof of if part.* Let us assume that $K$ is an unknotting number one knot. Then as indicated in Figure 4, we can choose a band $b$ so that the band sum with $b$ corresponds to a crossing change which converts $K$ into a trivial knot. Thus $L_b$ is a trivial knot.

*Proof of only if part.* The idea of a proof of the only if part of Theorem (1.4) is showing that the band sum producing a trivial knot is actually a crossing change converting $K$ into a trivial knot.

Let $b$ be a band such that $L_b$ is unknotted. Then since $L$ is a composite link $K \# H$, Eudave-Muñoz [1], Theorem 3, has shown:

**Lemma (2.1) (Eudave-Muñoz [1]).** There exists a decomposing 2-sphere $S$ intersecting $L$ transversely in two points with the following properties.

1. Neither of the 3-balls bounded by $S$ intersects $L$ in a single unknotted spanning arc.
2. $S$ crosses the band $b$ in a single arc parallel to $b(0,1] \times \{\frac{1}{2}\}$.

**Claim (2.2).** The 2-sphere $S$ gives a decomposition of $L$ as $K \# H$. 

Proof. It is easy to see that the two points in $S \cap L$ belong to $K$ and the other component $c$, a meridian of $K$, is entirely contained in a 3-ball, say $B$, bounded by $S$. Then $K \cap B'$, where $B'$ is the opposite side of $S$ in $S^3$, is a knotted spanning arc. Since $K$ is prime, $K \cap B$ is an unknotted spanning arc. Hence $S$ gives the required decomposition.

Using a result of Hirasawa and Shimokawa [3], we put further restriction on a position of the band $b$. To apply [3], Theorem 1.6, choose an orientation of $K$ arbitrarily and then choose an orientation of $c$ so that $L$ and $L_b$ have coherent orientations except for the band $b$.

Then [3], Theorem 1.6, asserts:

**Lemma (2.3) (Hirasawa-Shimokawa [3]).** There exists a minimal genus Seifert surface $F$ for the link $L$ which contains the band $b$.

Let $D$ be a disk bounded by $c$ which intersects $K$ exactly once.

**Lemma (2.4).** We may assume, if necessary after sliding the band $b$ along $L$, that $b$ does not intersect the interior of $D$, i.e. $b \cap D = b([0, 1] \times \{1\})$.

![Figure 5. Position of the band $b$](image)

**Proof.** Recall first that $L$ intersects the 2-sphere $S$ transversely in two points and that the band $b$ intersects $S$ transversely in a single arc $b([0, 1] \times \{\frac{1}{2}\})$. As usual we can isotope $F$ keeping $S, b, L$ invariant so that $F$ intersects $S$ transversely. Then $F \cap S$ consists of a single arc component and circle components. By an innermost disk argument we can eliminate the circle components (keeping $S, b, L$ invariant) to obtain a minimal genus Seifert surface $F$ intersecting $S$ transversely in a single arc which contains the arc $b \cap S$. Since $S$ crosses the band $b$ in a single arc parallel to $b([0, 1] \times \{\frac{1}{2}\})$ (Lemma (1.3)), $b$ crosses the arc $F \cap S$ just once.

Cutting the minimal genus Seifert surface $F$ along the arc $F \cap S$, we obtain a minimal genus Seifert surface $F_H$ of the Hopf link $H$, which is an annulus. Since $F_H$ is an annulus, we can slide the band $b$ along $c$ so that $b$ does not intersects the interior of $D$ as desired, see Figure 5.

It follows from Lemma (2.4) that the band sum operation can be regarded as a crossing change. Thus the knot $K$ becomes a trivial knot after the single crossing change, and hence $K$ has unknotting number one as desired.

This completes a proof of the only if part of Theorem (1.4).
3. Proof of Theorem (1.5).

Let $L$ be a 2-component link $T_{p,q} \cup c$ consisting of a torus knot $T_{p,q}$ ($|p| > q \geq 2$) and its meridian $c$. It is known that the unknotting number of a torus knot $T_{p,q}$ ($|p| > q \geq 2$) is $\frac{(|p| - 1)}{2(q - 1)}$ ([5]), hence Theorem (1.4) shows that $L_b$ is a trivial knot for some band $b$ if and only if $(p, q) = (\pm 3, 2)$.

Let us prove the uniqueness of such a trivializing band. Suppose that $K = T_{\pm 3, 2}$ and $b$ is a band connecting $K$ and $c$ such that $L_b$ is a trivial knot. Let $D$ be a disk bounded by $c$ intersecting $K$ exactly once. From Lemma (2.4), we may assume that $b$ does not intersect the interior of $D$. Following Kobayashi [4], we call a disk $\Delta$ intersecting $K$ in two points of opposite orientations a crossing disk for $K$. Now we associate a crossing disk $\Delta_b$ to the band $b$. Denote the core $b(\frac{1}{2} \times [0, 1]) \subset b$ by $\tau$. Extending the 1-complex $\tau \cup D$ to obtain a disk $\Delta_b$ so that (i) the linking number between $K$ and $\partial \Delta_b$ is zero, and (ii) $\Delta_b \cap \text{int}b = \text{int}\tau$, see Figure 6. Thus a band $b$ determines a crossing disk $\Delta_b$ with a surgery slope $\varepsilon(= \pm 1)$ on $\partial \Delta_b$ so that the band sum is realized by the surgery on $\partial \Delta_b$.

Conversely a crossing disk $\Delta$ with a meridian $c$ (or $D$) and a surgery slope $\varepsilon$ on $\partial \Delta$ determines a band $b$ uniquely.

Since $K = T_{\pm 3, 2}$ has unknotting number one, we can choose a band $b$ so that $L_b$ is a trivial knot. Let $\Delta_b$ be a crossing disk associated to the band $b$. Then using [4] we have a minimal genus Seifert surface $F$ for $K$ such that (1) $F$ is obtained from two Hopf bands by a plumbing along a disk, and (2) $K \cup \Delta_b$ has a position given by Figure 7 up to isotopy; a presentation of a trefoil knot $T_{\pm 3, 2}$ as the plumbing of two Hopf bands is unique up to isotopy. (In [4] Kobayashi describes a position of $\partial \Delta_b$, but his proof shows that we have the same conclusion for the crossing disk $\Delta_b$.)

For the crossing disk $\Delta_b$ we have two possibilities for the position of the disk $D$ as in Figure 7. In Figure 7 we indicate cores $\tau$ of two possible bands $b_1$ (in the left) and $b_2$ (in the right) depending on the positions of $D$. It is easy to see that there is an isotopy of $S^3$ deforming $L \cup b_1$ to $L \cup b_2$.

Furthermore, we can easily isotope $L \cup b_1$ to $L \cup b$ given in Figure 4 as required.

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Figure 7

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