Bol. Soc. Mat. Mexicana (3) Vol. 15, 2009

BAND SUM OPERATIONS YIELDING TRIVIAL KNOTS

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ABSTRACT. Let L be a 2-component link in the 3-sphere S^3 consisting of a knot K and its meridian c. Let $b\colon [0,1]\times[0,1]\to S^3$ be an embedding such that $b([0,1]\times[0,1])\cap K = b([0,1]\times\{0\})$ and $b([0,1]\times[0,1])\cap c = b([0,1]\times\{1\})$. Then we obtain a knot L_b by replacing $b([0,1]\times\{0,1\})$ in L with $b(\{0,1\}\times[0,1])$. We call L_b a *band sum* of L with the band b. If K is a trivial knot, i.e. L is a Hopf link, then Thompson has proved that only obvious band can create a trivial knot L_b. In the present paper we will show that K is a nontrivial knot and L_b is a trivial knot for some band b if and only if K has unknotting number one. As a particular case, if K is a torus knot, we determine the band b with L_b a trivial knot.

1. Introduction

Let $L = K_1 \cup K_2$ be a 2-component link in the 3-sphere S^3. Let $b\colon [0,1]\times[0,1]\to S^3$ be an embedding such that $b([0,1]\times[0,1])\cap K_1 = b([0,1]\times\{0\})$ and $b([0,1]\times[0,1])\cap K_2 = b([0,1]\times\{1\})$. Then we obtain a knot L_b by replacing $b([0,1]\times\{0,1\})$ in L with $b(\{0,1\}\times[0,1])$, see Figure 1. We call L_b a *band sum* of L with the band b. In the following, for simplicity, we use the same symbol b to denote the image $b([0,1]\times[0,1])$.

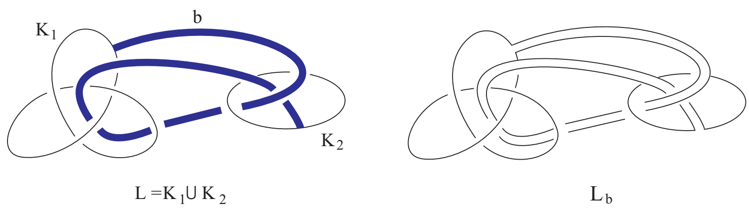

Figure 1. Band sum operation

It follows from [6] and [8] that we completely understand when we can obtain a trivial knot from a split link, i.e. there is a 2-sphere (called *splitting sphere*) in S^3 which separates K_1 and K_2, by band sum operation.

THEOREM (1.1) (Scharlemann [6], Thompson [8]). *Let L be a 2-component split link. If a band sum L_b is a trivial knot in S^3, then both K_1 and K_2 are unknotted and the band b is trivial, i.e. there is a splitting 2-sphere S so that $b\cap S$ consists of a single arc.*

2000 *Mathematics Subject Classification*: Primary 57M25.

Keywords and phrases: band sum, trivial knot.

The second named author was supported in part by Grant-in-Aid for Scientific Research (No. 17540097), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Now we start with a 2-component link obtained from a split link by a single crossing change in different components. To make precise we say that a 2-component link $L = K_1 \cup K_2$ is a *Hopf sum* of K_1 and K_2 and denote by $K_1 \cup_H K_2$ if L is obtained from K_1, K_2 and the Hopf link by conneced sum operation in a suitable way, see Figure 2. This operation depends on the orientations of K_1, K_2 and the Hopf link, but in the following such an ambiguity is irrelevant.

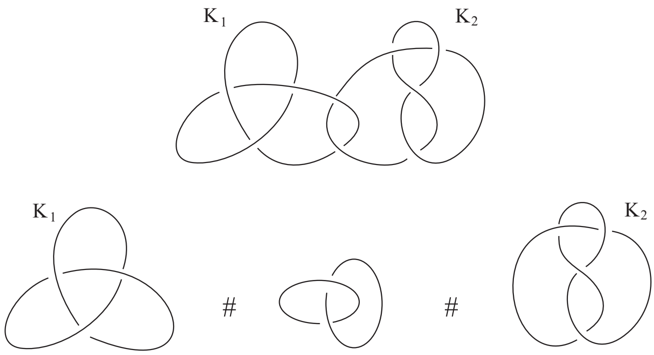

Figure 2. Hopf sum $K_1 \cup_H K_2$

In the case where both K_1 and K_2 are unknotted, i.e. L is a Hopf link H, Thompson [9], Corollary 3, has proved the following. See [3] for an alternate proof and a generalization where L is a $(2, 2p)$-torus link.

THEOREM (1.2) (Thompson). *Let L be a Hopf link. Then L_b is a trivial knot if and only if $L \cup b(\{\frac{1}{2}\} \times [0, 1])$ has a planar projection with exactly two crossings and b is untwisted or half-twisted.*

Furthermore, following Eudave-Muñoz [1], Corollary 2, we have:

THEOREM (1.3) (Eudave-Muñoz). *Let L be a Hopf sum $K_1 \cup_H K_2$ of K_1 and K_2.*

(1) *If both K_1 and K_2 are knotted, then L_b is knotted for any band b.*

(2) *If K_1 is a composite knot and K_2 is a trivial knot, then L_b is knotted for any band b.*

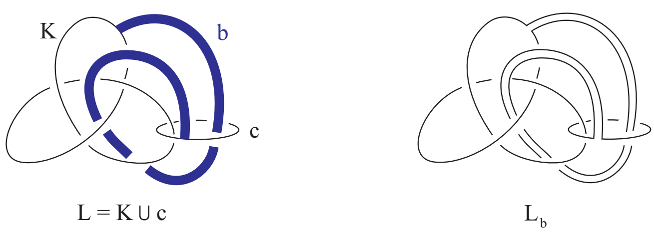

Figure 3. Band sum of a Hopf sum

In the present note we will consider the remaining case: K_1 is a (nontrivial) prime knot and K_2 is a trivial knot. Then L consists of the prime knot $K_1 = K$ and its meridian $K_2 = c$, see Figure 3.

THEOREM (1.4). *Let L be a 2-component link $K \cup c$ consisting of a prime knot K and its meridian c. Then a band sum L_b is a trivial knot for some band b if and only if K has unknotting number one.*

Theorem (1.4), together with a result of Kobayashi [4] ([7]), we have the following result which is motivated by a study of Seifert surgery on knots [2].

THEOREM (1.5). *Let L be a 2-component link $T_{p,q} \cup c$ consisting of a (p, q)-torus knot $T_{p,q}$ ($|p| > q \geq 2$) and its meridian c. If a band sum L_b is a trivial knot, then $(p, q) = (\pm 3, 2)$ and the band b is given by Figure 4 up to isotopy in S^3; the isotopy is not necessarily leaving the link L invariant.*

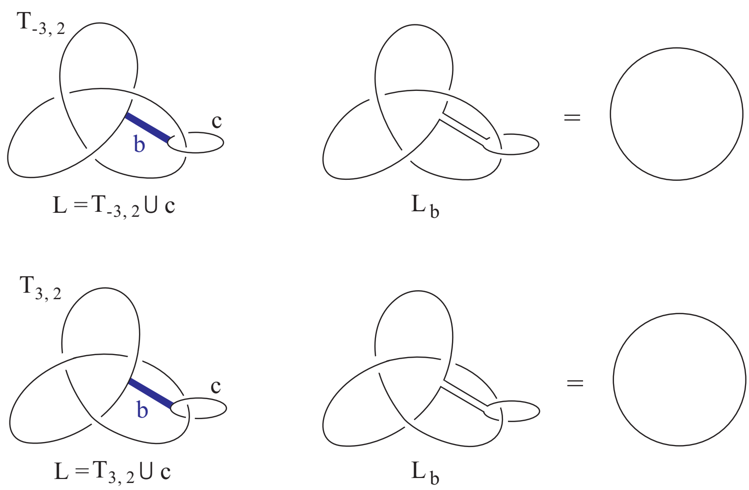

Figure 4. Trivializing band for $T_{p,q} \cup c$

2. Proof of Theorem (1.4)

Proof of if part. Let us assume that K is an unknotting number one knot. Then as indicated in Figure 4, we can choose a band b so that the band sum with b corresponds to a crossing change which converts K into a trivial knot. Thus L_b is a trivial knot.

Proof of only if part. The idea of a proof of the only if part of Theorem (1.4) is showing that the band sum producing a trivial knot is actually a crossing change converting K into a trivial knot.

Let b be a band such that L_b is unknotted. Then since L is a composite link $K\#H$, Eudave-Muñoz [1], Theorem 3, has shown:

LEMMA (2.1) (Eudave-Muñoz [1]). *There exists a decomposing 2-sphere S intersecting L transversely in two points with the following properties.*

(1) *Neither of the 3-balls bounded by S intersects L in a single unknotted spanning arc.*

(2) *S crosses the band b in a single arc parallel to $b([0, 1] \times \{\frac{1}{2}\})$.*

CLAIM (2.2). *The 2-sphere S gives a decomposition of L as $K\#H$.*

Proof. It is easy to see that the two points in $S \cap L$ belong to K and the other component c, a meridian of K, is entirely contained in a 3-ball, say B, bounded by S. Then $K \cap B'$, where B' is the opposite side of S in S^3, is a knotted spanning arc. Since K is prime, $K \cap B$ is an unknotted spanning arc. Hence S gives the required decomposition. □

Using a result of Hirasawa and Shimokawa [3], we put further restriction on a position of the band b. To apply [3], Theorem 1.6, choose an orientation of K arbitrarily and then choose an orientation of c so that L and L_b have coherent orientations except for the band b.

Then [3], Theorem 1.6, asserts:

LEMMA (2.3) (Hirasawa-Shimokawa [3]). *There exists a minimal genus Seifert surface F for the link L which contains the band b.*

Let D be a disk bounded by c which intersects K exactly once.

LEMMA (2.4). *We may assume, if necessary after sliding the band b along L, that b does not intersect the interior of D, i.e. $b \cap D = b([0, 1] \times \{1\})$.*

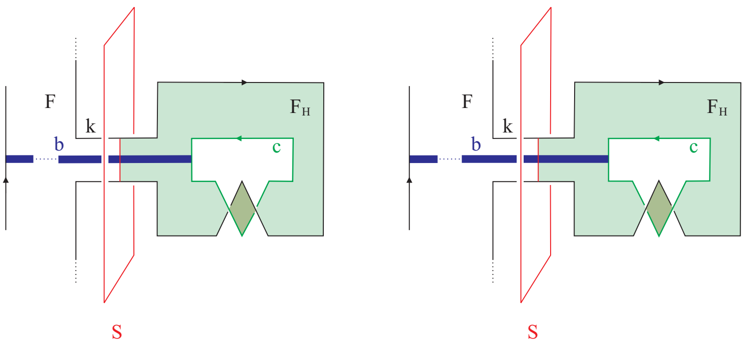

Figure 5. Position of the band b

Proof. Recall first that L intersects the 2-sphere S transversely in two points and that the band b intersects S transversely in a single arc $b([0, 1] \times \{\frac{1}{2}\})$. As usual we can isotope F keeping S, b, L invariant so that F intersects S transversely. Then $F \cap S$ consists of a single arc componet and circle components. By an innermost disk argument we can eliminate the circle components (keeping S, b, L invariant) to obtain a minimal genus Seifert surface F intersecting S transversely in a single arc which contains the arc $b \cap S$. Since S crosses the band b in a single arc parallel to $b([0, 1] \times \{\frac{1}{2}\})$ (Lemma (1.3)), b crosses the arc $F \cap S$ just once.

Cutting the minimal genus Seifert surface F along the arc $F \cap S$, we obtain a minimal genus Seifert surface F_H of the Hopf link H, which is an annulus. Since F_H is an annulus, we can slide the band b along c so that b does not intersects the interior of D as desired, see Figure 5. □

It follows from Lemma (2.4) that the band sum operation can be regarded as a crossing change. Thus the knot K becomes a trivial knot after the single crossing change, and hence K has unknotting number one as desired.

This completes a proof of the only if part of Theorem (1.4).

3. Proof of Theorem (1.5).

Let L be a 2-component link $T_{p,q} \cup c$ consisting of a torus knot $T_{p,q}$ ($|p| > q \geq 2$) and its meridian c. It is known that the unknotting number of a torus knot $T_{p,q}$ ($|p| > q \geq 2$) is $\frac{(|p|-1)(q-1)}{2}$ ([5]), hence Theorem (1.4) shows that L_b is a trivial knot for some band b if and only if $(p, q) = (\pm 3, 2)$.

Let us prove the uniqueness of such a trivializing band. Suppose that $K = T_{\pm 3,2}$ and b is a band connecting K and c such that L_b is a trivial knot. Let D be a disk bounded by c intersecting K exactly once. From Lemma (2.4), we may assume that b does not intersect the interior of D. Following Kobayashi [4], we call a disk Δ intersecting K in two points of opposite orientations a *crossing disk* for K. Now we associate a crossing disk Δ_b to the band b. Denote the core $b(\{\frac{1}{2}\} \times [0, 1]) \subset b$ by τ. Extending the 1-complex $\tau \cup D$ to obtain a disk Δ_b so that (i) the linking number between K and $\partial\Delta_b$ is zero, and (ii) $\Delta_b \cap \mathrm{int} b = \mathrm{int}\tau$, see Figure 6. Thus a band b determines a crossing disk Δ_b with a surgery slope $\varepsilon(= \pm 1)$ on $\partial\Delta_b$ so that the band sum is realized by the surgery on $\partial\Delta_b$.

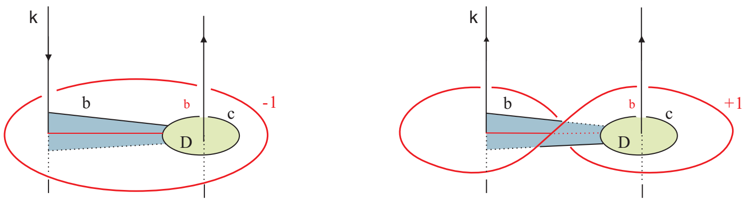

Figure 6. In the left, L_b is the result of (-1)-surgery on $\partial\Delta_b$; in the right, L_b is the result of $(+1)$-surgery on $\partial\Delta_b$

Conversely a crossing disk Δ with a meridian c (or D) and a surgery slope ε on $\partial\Delta$ determines a band b uniquely.

Since $K = T_{\pm 3,2}$ has unknotting number one, we can choose a band b so that L_b is a trivial knot. Let Δ_b be a crossing disk associated to the band b. Then using [4] we have a minimal genus Seifert surface F for K such that (1) F is obtained from two Hopf bands by a plumbing along a disk, and (2) $K \cup \Delta_b$ has a position given by Figure 7 up to isotopy; a presentation of a trefoil knot $T_{\pm 3,2}$ as the plumbing of two Hopf bands is unique up to isotopy. (In [4] Kobayashi describes a position of $\partial\Delta_b$, but his proof shows that we have the same conclusion for the crossing disk Δ_b.)

For the crossing disk Δ_b we have two possibilities for the position of the disk D as in Figure 7. In Figure 7 we indicate cores τ of two possible bands b_1 (in the left) and b_2 (in the right) depending on the positions of D. It is easy to see that there is an isotopy of S^3 deforming $L \cup b_1$ to $L \cup b_2$.

Furthermore, we can easily isotope $L \cup b_1$ to $L \cup b$ given in Figure 4 as required.

Acknowledgements

We would like to thank Masaharu Ishikawa and Koya Shimokawa for helpful discussions. We would also like to thank the referee for careful reading and useful comments.

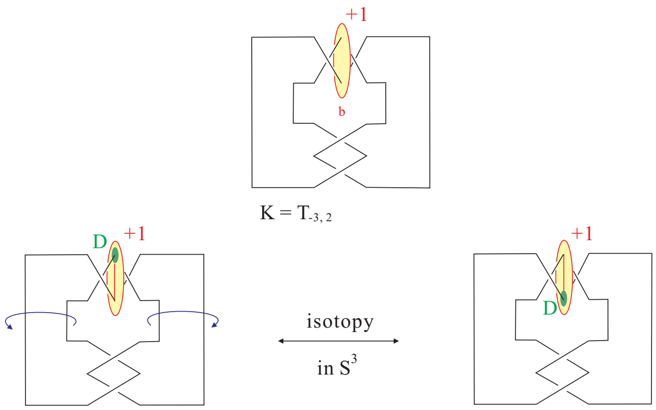

Figure 7

Received October 19, 2007

Final version received September 26, 2008

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