NONTRIVIAL ELEMENTS IN A KNOT GROUP WHICH ARE TRIVIALIZED BY DEHN FILLINGS

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Abstract. Let K be a nontrivial knot in S^3 with the exterior $E(K)$, and $\gamma \in G(K) = \pi_1(E(K), *)$ a slope element represented by an essential simple closed curve on $\partial E(K)$ with base point $* \in \partial E(K)$. Since the normal closure $\langle\langle \gamma \rangle\rangle$ of γ in $G(K)$ coincides with that of γ^{-1}, and γ and γ^{-1} correspond to a slope $r \in \mathbb{Q} \cup \{\infty\}$, we write $\langle\langle r \rangle\rangle = \langle\langle \gamma \rangle\rangle$. The normal closure $\langle\langle r \rangle\rangle$ describes elements which are trivialized by r–Dehn filling of $E(K)$. In this article, we prove that $\langle\langle r_1 \rangle\rangle = \langle\langle r_2 \rangle\rangle$ if and only if $r_1 = r_2$, and for a given finite family of slopes $\mathcal{S} = \{r_1, \dots, r_n\}$, the intersection $\langle\langle r_1 \rangle\rangle \cap \cdots \cap \langle\langle r_n \rangle\rangle$ contains infinitely many elements except when K is a (p, q)–torus knot and $pq \in \mathcal{S}$. We also investigate inclusion relation among normal closures of slope elements.

1. Introduction

Geometric aspects of Dehn fillings such as destroying and creating essential surfaces have been extensively studied by many authors; see survey articles [15, 16, 17, 18] and references therein. In the present article we focus on a group theoretic aspect of Dehn fillings. Let K be a nontrivial knot in S^3 with its exterior $E(K)$. Then by the loop theorem [45] the inclusion map $i : \partial E(K) \to E(K)$ induces a monomorphism $i_* : \pi_1(\partial E(K), *) \to \pi_1(E(K), *)$, where we choose a base point $*$ in $\partial E(K)$. We denote the *knot group* $\pi_1(E(K), *)$ by $G(K)$ and its *peripheral subgroup* $i_*(\pi_1(\partial E(K), *))$ by $P(K)$. A *slope element* in $G(K)$ is a primitive element γ in $P(K) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is represented by an essential oriented simple closed curve on $\partial E(K)$ with base point $*$. Denote by $\langle\langle \gamma \rangle\rangle$ the normal closure of γ in $G(K)$. Taking a standard meridian-longitude pair (μ, λ) of K, each slope element γ is expressed as $\mu^m \lambda^n$ for some relatively prime integers m, n. As usual we use the term *slope* to mean the isotopy class of an essential unoriented simple closed curve on $\partial E(K)$. A slope element γ and its inverse γ^{-1} give the same

The first named author has been partially supported by JSPS KAKENHI Grant Number JP15K17540 and JP16H02145.

The second named author has been partially supported by JSPS KAKENHI Grant Number JP26400099 and Joint Research Grant of Institute of Natural Sciences at Nihon University for 2017.

The third named author has been partially supported by JSPS KAKENHI Grant Number JP16K05149.

2010 *Mathematics Subject Classification.* Primary 57M05, 57M25

normal subgroup $\langle\langle \gamma \rangle\rangle = \langle\langle \gamma^{-1} \rangle\rangle$, and by forgetting orientations, they correspond to the same slope, which may be identified with $m/n \in \mathbb{Q} \cup \{\infty\}$. So in the following we denote $\langle\langle \gamma \rangle\rangle = \langle\langle \gamma^{-1} \rangle\rangle$ by $\langle\langle m/n \rangle\rangle$. Thus each slope m/n defines the normal subgroup $\langle\langle m/n \rangle\rangle \subset G(K)$, which will be referred to as the *normal closure of the slope* m/n for simplicity. A slope r is *trivial* if $r = \infty$, i.e. r is represented by a meridian of K. In what follows, we abbreviate the base point for simplicity.

Denote by $K(r)$ the 3–manifold obtained from $E(K)$ by r–Dehn filling. Then we have the following short exact sequence which relates $G(K)$, $\langle\langle r \rangle\rangle$ and $\pi_1(K(r))$.

$$\{1\} \to \langle\langle r \rangle\rangle \to G(K) \to G(K)/\langle\langle r \rangle\rangle = \pi_1(K(r)) \to \{1\},$$

and thus

$$\langle\langle r \rangle\rangle = \{g \in G(K) \mid g \text{ becomes trivial in } \pi_1(K(r)\}.$$

Recall that a group G possesses the *Magnus property*, if whenever two elements u, v of G have the same normal closure, then u is conjugate to v or v^{-1}. Magnus [34] established this property for free groups, and recently [3, 4, 11, 24] prove the fundamental groups of closed surfaces have this property. However, in general, knot groups do not satisfy this property; see [8, 47, 48] for details. With respect to Dehn fillings, the above observation leads us to introduce:

Definition 1.1 (**peripheral Magnus property**)**.** Let K be a nontrivial knot in S^3. We say that the knot group $G(K)$ has the *peripheral Magnus property* if $\langle\langle r \rangle\rangle = \langle\langle r' \rangle\rangle$ implies $r = r'$ for two slopes r and r'.

Property P [32] says that $\langle\langle r \rangle\rangle = \langle\langle \infty \rangle\rangle = G(K)$ if and only if $r = \infty$. We first establish every nontrivial knot group has this property.

Theorem 1.2. *For any nontrivial knot K, the knot group $G(K)$ satisfies the peripheral Magnus property, namely $\langle\langle r \rangle\rangle = \langle\langle r' \rangle\rangle$ if and only if $r = r'$.*

When K is a prime, non-amphicheiral knot, we will prove a slightly stronger version of Theorem 1.2; see Theorem 3.5.

Theorem 1.2 says that there is a one to one correspondence between the set of slopes, which is identified with $\mathbb{Q} \cup \{\infty\}$, and the set of normal closures of slopes.

Next we investigate for which slope $r \in \mathbb{Q} \cup \{\infty\}$, its normal closure $\langle\langle r \rangle\rangle$ is finitely generated. There are two obvious situations where $\langle\langle r \rangle\rangle$ is finitely generated.

- If K has a *finite surgery slope* r, i.e. r–surgery on K yields a 3–manifold with finite fundamental group, then $\langle\langle r \rangle\rangle$ is finitely generated. (See the proof of Theorem 1.3.)
- If K is a torus knot $T_{p,q}$, then $\langle\langle pq \rangle\rangle$ is an infinite cyclic normal subgroup of $G(K)$, hence finitely generated.

Theorem 1.3 below classifies slopes whose normal closures are finitely generated.

Theorem 1.3. *Let K be a nontrivial knot. The normal closure $\langle\langle r \rangle\rangle$ is finitely generated if and only if r is a finite surgery slope, or K is a torus knot $T_{p,q}$ and $r = pq$.*

Thus generically, normal closures of slopes are infinitely generated, hence each Dehn filling trivializes an infinitely generated subgroup of $G(K)$. So it seems interesting to ask: For how many slopes of K do their normal closures intersect nontrivially? Furthermore, if the intersection is nontrivial, how big is this subgroup?

Theorem 1.4. *Let K be a nontrivial knot in S^3, and let $\{r_1, \dots, r_n\}$ $(n \geq 2)$ be any finite family of slopes of K. If K is a torus knot $T_{p,q}$, we assume that $pq \notin \{r_1, \dots, r_n\}$. Then $\langle\langle r_1 \rangle\rangle \cap \cdots \cap \langle\langle r_n \rangle\rangle$ is nontrivial. Moreover, this subgroup is finitely generated if and only if all the r_i are finite surgery slopes.*

Remark 1.5.
- (i) *If $K = T_{p,q}$ and $pq \in \{r_1, \dots, r_n\}$, then generically $\langle\langle r_1 \rangle\rangle \cap \cdots \cap \langle\langle r_n \rangle\rangle = \{1\}$; see Proposition 5.4.*
- (ii) *As the result below shows, the finiteness of a family of slopes in Theorem 1.4 is essential.*

Theorem 1.6 ([26])**.** *Let K be a hyperbolic knot in S^3. Then $\langle\langle r_1 \rangle\rangle \cap \langle\langle r_2 \rangle\rangle \cap \cdots = \{1\}$ for any infinite family of slopes $\{r_1, r_2, \dots\}$.*

Combining Theorems 1.4 and 1.6 we have:

Corollary 1.7. *Let K be a hyperbolic knot in S^3. For any infinite family of slopes, their normal closures intersect trivially, while for any finite subfamily, the intersection of their normal closures contains infinitely many elements.*

This has the following interpretation from a viewpoint of Dehn fillings.

Corollary 1.8. *Let K be a hyperbolic knot in S^3. Then for any nontrivial element $g \in G(K)$, there are only finitely many Dehn fillings of $E(K)$ which trivialize g, while for any finitely many Dehn fillings, there are infinitely many nontrivial elements which become trivial by each of these Dehn fillings.*

Remark 1.9.
- (i) *Let K be a nontrivial torus knot $T_{p,q}$. Then $\cap_{n \in \mathbb{Z}} \langle\langle (pqn \pm 1)/n \rangle\rangle = [G(K), G(K)]$, which is the free group of rank $(p-1)(q-1)/2$. See Proposition 6.5.*
- (ii) *For satellite knots, it is still open, but we expect $\langle\langle r_1 \rangle\rangle \cap \langle\langle r_2 \rangle\rangle \cap \cdots = \{1\}$ for any infinite family of slopes $\{r_1, r_2, \dots\}$. See* [27] *for further discussion.*

Let us turn to inclusion relations among normal closures of slopes. Since $\langle\langle \infty \rangle\rangle = G(K)$, $\langle\langle r \rangle\rangle \subset \langle\langle \infty \rangle\rangle$ for any slope r.

Theorem 1.10. *Let K be a non-torus knot in S^3. If $\langle\langle r_1 \rangle\rangle \supset \cdots \supset \langle\langle r_n \rangle\rangle$ for mutually distinct slopes $r_1, r_2, \dots, r_n \in \mathbb{Q}$, then $n \le 2$. In particular, there is no infinite descending chain nor ascending chain of normal closures of slopes.*

On the contrary, for torus knots we have:

Theorem 1.11. *Let K be a torus knot $T_{p,q}$ $(p > q \ge 2)$.*

(i) *There is no infinite ascending chain $\langle\langle r_1 \rangle\rangle \subset \langle\langle r_2 \rangle\rangle \subset \langle\langle r_3 \rangle\rangle \subset \cdots$.*

(ii) *For each finite surgery slope r, there exists an infinite descending chain*

$$\langle\langle r \rangle\rangle \supset \langle\langle r_1 \rangle\rangle \supset \langle\langle r_2 \rangle\rangle \supset \cdots .$$

2. Inclusions between two normal closures of slopes

In this section we study inclusions between two normal closures of slopes. We say that a slope r is a *reducing surgery slope* if $K(r)$ is a reducible 3-manifold.

Lemma 2.1. *Let K be a nontrivial knot in S^3 with meridian μ. If $\mu^a \in \langle\langle r \rangle\rangle$ for some integer $a \neq 0$, then r is a finite surgery slope or a reducing surgery slope.*

Proof. Without loss of generality we may assume $a > 0$. If $a = 1$, then $\mu \in \langle\langle r \rangle\rangle$ and $\pi_1(K(r)) = G(K)/\langle\langle r \rangle\rangle = \{1\}$, and thus r is a finite surgery slope. (Actually, Property P [32] implies $r = \infty$.) So in the following we assume $a \ge 2$. Since $\mu^a \in \langle\langle r \rangle\rangle$, $\langle\langle \mu^a \rangle\rangle \subset \langle\langle r \rangle\rangle$ and we have the canonical epimorphism $\varphi : G(K)/\langle\langle \mu^a \rangle\rangle \to G(K)/\langle\langle r \rangle\rangle$. Note that $(\varphi(\mu))^a = \varphi(\mu^a) = 1$ in $G(K)/\langle\langle r \rangle\rangle$. If $\varphi(\mu) = 1 \in G(K)/\langle\langle r \rangle\rangle$, then $\varphi(\mu) = \mu \in \langle\langle r \rangle\rangle$ and as above $r = \infty$. Thus we may assume $\varphi(\mu) \neq 1$, i.e. it is a nontrivial torsion element in $G(K)/\langle\langle r \rangle\rangle$. Recall that an irreducible 3–manifold M with infinite fundamental group is aspherical [1, p.48 (C.1)] and hence $\pi_1(M)$ has no torsion element [22]. Hence $\pi_1(K(r))$ is finite or $K(r)$ must be a reducible manifold. Accordingly r is a finite surgery slope or a reducing surgery slope. □

Lemma 2.2. *Let K be a nontrivial knot in S^3 with meridian μ, and r a nontrivial slope. If $\mu^a \in \langle\langle r \rangle\rangle$ for some integer $a \neq 0$, then r is not a reducing surgery slope.*

Proof. Assume to the contrary that r is a reducing surgery slope. As in the proof of Lemma 2.1 $G(K)/\langle\langle r \rangle\rangle$ has a nontrivial torsion element, hence $K(r) \neq S^2 \times S^1$ and thus $K(r)$ is a connected sum of two closed 3–manifolds other than S^3. (In general, the result of a surgery on a nontrivial knot is not $S^2 \times S^1$ [12].) By the Poincaré conjecture, they have nontrivial fundamental groups, and $G(K)/\langle\langle r \rangle\rangle = A * B$ for some nontrivial groups A and B.

As we have seen in the proof of Lemma 2.1, $\varphi(\mu)$, the image of μ under the canonical epimorphism $\varphi : G(K)/\langle\langle \mu^a \rangle\rangle \to G(K)/\langle\langle r \rangle\rangle$ is a nontrivial torsion element in $A * B$. By [35, Corollary 4.1.4], a nontrivial torsion element in a free product $A*B$ is conjugate to a torsion element of A or B. Thus we may assume that there exists $g \in A*B$ such that $g\varphi(\mu)g^{-1} \in A$.

On the other hand, $A*B$ is normally generated by $\varphi(\mu)$ since $G(K)$ is normally generated by μ. This implies that $A*B$ is normally generated by an element $g\varphi(\mu)g^{-1} \in A$. In particular, the normal closure $\langle\langle A \rangle\rangle$ of A in $A*B$ is equal to $A*B$, and $(A*B)/\langle\langle A \rangle\rangle = \{1\}$. However, $(A*B)/\langle\langle A \rangle\rangle = B \neq \{1\}$ ([35, p.194]). This is a contradiction. □

Combine Lemmas 2.1 and 2.2 to obtain the following result which asserts that inclusions among normal closures of slopes are quite limited.

Proposition 2.3. *Let K be a nontrivial knot in S^3. Assume that $\langle\langle r \rangle\rangle \supset \langle\langle r' \rangle\rangle$ for distinct slopes r and r'. Then r is a finite surgery slope.*

Proof. If $r = \infty$, there is nothing to prove. Hence we assume $r \neq \infty$. Write $r = m/n$ and $r' = m'/n'$. By the assumption $\mu^{m'}\lambda^{n'} \in \langle\langle m/n \rangle\rangle$, and hence

$$\mu^{-mn'+nm'} = (\mu^m\lambda^n)^{-n'}(\mu^{m'}\lambda^{n'})^n \in \langle\langle m/n \rangle\rangle.$$

Since r and r' are distinct slopes, $-mn'+nm' \neq 0$. Then Lemma 2.1 shows that r is a finite surgery slope or a reducing surgery slope. However, the latter case cannot happen by Lemma 2.2. □

Proposition 2.4. *Let K be a non-torus knot in S^3. Assume that $\langle\langle r \rangle\rangle \supset \langle\langle r' \rangle\rangle$ for distinct nontrivial slopes r and r'. Then r' is not a finite surgery slope.*

Proof. By the assumption and Proposition 2.3, r is a nontrivial finite surgery slope. Assume to the contrary that r' is also a finite surgery slope. Write $r = m/n$ and $r' = m'/n'$ with $m, m' > 0$. Since $\langle\langle m/n \rangle\rangle \supset \langle\langle m'/n' \rangle\rangle$, we have a canonical epimorphism from $G(K)/\langle\langle m'/n' \rangle\rangle$ to $G(K)/\langle\langle m/n \rangle\rangle$, which induces an epimorphism

$$\mathbb{Z}_{m'} \cong H_1(K(m'/n')) \to H_1(K(m/n)) \cong \mathbb{Z}_m.$$

This then implies that $m' \geq m$ and m' is a multiple of m.

By the assumption K is a hyperbolic knot or a satellite knot. Furthermore, in the latter case, since K admits a nontrivial finite surgery, K is a (p,q)–cable of a torus knot $T_{p',q'}$, where $|p| \geq 2$ [5].

Case 1. K is a hyperbolic knot.

Recall that the distance between finite surgery slopes of a hyperbolic knot is at most two [42]. Hence $|mn' - nm'| \leq 2$. Since m' is a multiple of m, the inequality $|mn' - nm'| \leq 2$ implies $m = 1, 2$. Since $n \neq 0$, we have $|\frac{m}{n}| \leq 2$. A finite surgery is also an L-space surgery, so by [44, Corollary 1.4], $2 \geq |\frac{m}{n}| \geq 2g(K) - 1$, where $g(K)$ is the genus of K. This implies $g(K) = 1$. Since a knot admitting an L-space surgery is fibered [40, 41, 14, 31], K is a trefoil knot $T_{3,2}$ (or $T_{-3,2}$) or the figure-eight knot. By assumption, K is not a torus knot, and hence K would be the figure-eight knot. However, the figure-eight knot has no nontrivial finite surgery, a contradiction.

Case 2. K is a (p, q)–cable of a torus knot $T_{p',q'}$.

Finite surgeries on iterated torus knots are classified by [2, Table 1]. For any cable of a torus knot which admits two finite surgeries m/n and m'/n', m' is not a multiple of m.

This completes a proof of Proposition 2.4. □

3. Peripheral Magnus property for knot groups

Now we are ready to prove the peripheral Magnus property for knot groups. We separate the proof into two cases depending upon K is a non-torus knot or K is a torus knot. Furthermore, in the latter case, we distinguish the specific case where K is a trefoil knot $T_{3,2}$ and $r = (18k + 9)/(3k + 1)$, $r' = (18k + 9)/(3k + 2)$ for technical reasons; see Proposition 3.1. These surgeries are only examples of surgeries along torus knots which give rise to (orientation reversingly) homeomorphic 3-manifolds with finite fundamental group [36]. Thus, the case treated in Proposition 3.1 gives us the most subtle situation.

Theorem 1.2. Any nontrivial knot satisfies the peripheral Magnus property, namely $\langle\langle r\rangle\rangle = \langle\langle r'\rangle\rangle$ if and only if $r = r'$.

Proof. The "if" part is obvious. Let us prove the "only if" part. Recall first that if $r = \infty$, then $\langle\langle r'\rangle\rangle = \langle\langle \infty\rangle\rangle$ can happen only when $r' = \infty$ by Property P [32]. If $r = 0$, then $\langle\langle r'\rangle\rangle = \langle\langle 0\rangle\rangle$ holds only if $r' = 0$ for homological reason. So in the following we assume r is neither ∞ nor 0. We divide the argument into two cases depending upon K is a torus knot or a non-torus knot.

Case 1. K is a non-torus knot.

Suppose for a contradiction that we have mutually distinct slopes r and r' which satisfy $\langle\langle r\rangle\rangle = \langle\langle r'\rangle\rangle$. Since $\langle\langle r\rangle\rangle \supset \langle\langle r'\rangle\rangle$, Propositions 2.3 and 2.4 show that r is a finite surgery slope and r' is not a finite surgery slope. On the other hand, since $\langle\langle r\rangle\rangle \subset \langle\langle r'\rangle\rangle$, Proposition 2.3 implies that r' is a finite surgery slope, a contradiction.

Case 2. K is a torus knot $T_{p,q}$.

Without loss of generality, we assume $p > q \geq 2$. Assume that $\langle\langle r\rangle\rangle = \langle\langle r'\rangle\rangle$ for mutually distinct slopes r and r'. By Proposition 2.3 r and r' are finite surgery slopes. Let us write $r = m/n$ and $r' = m'/n'$; we may assume $m, m' > 0$. Then $m = |H_1(K(m/n))| = |H_1(K(m'/n'))| = m'$. Recall that $T_{p,q}(m/n)$ has a Seifert fibration with base orbifold $S^2(p, q, |pqn - m|)$.

Assume first that $\pi_1(T_{p,q}(m/n))$ is cyclic, i.e. $|pqn - m| = 1$. Then since $\pi_1(T_{p,q}(m/n')) \cong \pi_1(T_{p,q}(m/n))$ is finite cyclic, we have $|pqn-m| = |pqn'-m| = 1$.

A simple computation shows that $|n - n'| = \frac{2}{pq}$, which is impossible, because $pq \geq 6$.

Suppose next that $\pi_1(T_{p,q}(m/n))$ is finite, but non-cyclic. Then $\{p, q, |pqn - m|\} = \{2, 2, A\}$ (where $A \geq 3$ is an odd integer), $\{2, 3, 3\}, \{2, 3, 4\}$ or $\{2, 3, 5\}$.

Subcase 1. Assume that $\{p, q, |pqn - m|\} = \{2, 2, A\}$. Then K is a torus knot $T_{2,A}$ and $|2An - m| = 2$. Since $\pi_1(T_{2,A}(m/n)) \cong \pi_1(T_{2,A}(m/n'))$, $T_{2,A}(m/n)$ and $T_{2,A}(m/n')$ are homeomorphic [1]. Thus $T_{A,2}(m/n')$ has a base orbifold $S^2(2, 2, A)$, where $|2An' - m| = 2$. By the assumption $n \neq n'$, and hence $|n - n'| = \frac{2}{A}$, which is an integer. This means $A = 1$ or 2, a contradiction.

Subcase 2. Assume that $\{p, q, |pqn - m|\} = \{2, 3, 3\}$. Then $K = T_{3,2}$ and $|6n - m| = 3$. Similarly, we have $|6n' - m| = 3$. Thus m is a multiple of 3 and since n and n' are coprime to m, neither n nor n' is divided by 3. Furthermore, the equalities $|6n - m| = |6n' - m| = 3$ gives $|n - n'| = 1$. Write $n = 3k + 1$ for some integer k. Then $n' = 3k + 2$ and $|6n - m| = |6n' - m| = 3$ is written as $|18k + 6 - m| = |18k + 12 - m| = 3$. Hence we have a unique solution $m = 18k + 9$. This gives $m/n = (18k + 9)/(3k + 1)$, $m/n' = (18k + 9)/(3k + 2)$. However, in this case $\langle\!\langle (18k + 9)/(3k + 1) \rangle\!\rangle \neq \langle\!\langle (18k + 9)/(3k + 2) \rangle\!\rangle$ by Proposition 3.1 below.

Subcase 3. Assume that $\{p, q, |pqn - m|\} = \{2, 3, 4\}$. Then we have two possibilities: $K = T_{3,2}$ and $|6n - m| = |6n' - m| = 4$, or $K = T_{4,3}$ and $|12n - m| = |12n' - m| = 2$. In the former case, $|n - n'| = \frac{4}{3} \notin \mathbb{Z}$, a contradiction. In the latter case, $|n - n'| = \frac{1}{3} \notin \mathbb{Z}$, a contradiction.

Subcase 4. Assume that $\{p, q, |pqn - m|\} = \{2, 3, 5\}$. Then we have three possibilities: $K = T_{3,2}$ and $|6n - m| = |6n' - m| = 5$, $K = T_{5,3}$ and $|15n - m| = |15n' - m| = 2$, or $K = T_{5,2}$ and $|10n - m| = |10n' - m| = 3$. In either case $|n - n'|$ cannot be an integer and we have a contradiction.

This completes a proof of Theorem 1.2. □

Proposition 3.1. *Let K be the trefoil knot $T_{3,2}$. In the knot group $G(K)$,*

$$\langle\!\langle (18k + 9)/(3k + 1) \rangle\!\rangle \neq \langle\!\langle (18k + 9)/(3k + 2) \rangle\!\rangle.$$

More precisely, $\langle\!\langle (18k+9)/(3k+1) \rangle\!\rangle \not\subset \langle\!\langle (18k+9)/(3k+2) \rangle\!\rangle$ *and* $\langle\!\langle (18k+9)/(3k+1) \rangle\!\rangle \not\supset \langle\!\langle (18k + 9)/(3k + 2) \rangle\!\rangle$.

Proof. Figure 3.1 shows that $K(r + 6)$ is a Seifert fibered manifold with Seifert invariant $S^2(1/r, -1/3, -1/2, 1)$.

Let us choose $r = 3/(3k + 1)$ so that $r + 6 = (18k + 9)/(3k + 1)$. Then $K((18k + 9)/(3k + 1))$ has a Seifert invariant:

$$S^2((3k + 1)/3, -1/3, -1/2, 1) = S^2(k; 1/3, -1/3, 1/2).$$

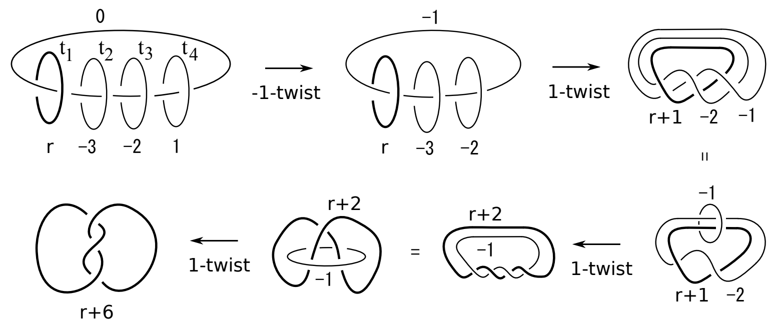

FIGURE 3.1. Surgery diagrams for $K(r+6)$

Similarly if we choose $r = -3/(3k+2)$, then $r+6 = (18k+9)/(3k+2)$, and $K((18k+9)/(3k+2))$ has a Seifert invariant:

$$S^2((-3k-2)/3, -1/3, -1/2, 1) = S^2(-k; 1/3, -1/3, -1/2).$$

This shows that $K((18k+9)/(3k+1))$ is orientation reversingly homeomorphic to $K((18k+9)/(3k+2))$ [36].

To obtain a presentation of their fundamental groups, we fix a section for the circle bundle $K((18k+9)/(3k+1)) - \bigcup_{i=1}^{4} N(t_i) = K((18k+9)/(3k+2)) - \bigcup_{i=1}^{4} N(t_i)$ arising from the top left picture of Figure 3.1, where $N(t_i)$ is a fibered tubular neighborhood. Note that t_1 is the surgery dual to K and t_4 is a regular fiber.

Then, with this section $\pi_1(K((18k+9)/(3k+1)))$ has a presentation:

$$\langle c_1, c_2, c_3, c_4, h \mid [c_i, h] = 1, c_1c_2c_3c_4 = 1, c_1^3h^{3k+1} = 1, c_2^3h^{-1} = 1, c_3^2h^{-1} = 1, c_4h = 1\rangle,$$

where c_i is represented by a meridian of t_i, a boundary of the section, and h is represented by a regular fiber.

Using the same bases $c_1, \dots, c_4, h$, $\pi_1(K((18k+9)/(3k+2)))$ has a presentation:

$$\langle c_1, c_2, c_3, c_4, h \mid [c_i, h] = 1, c_1c_2c_3c_4 = 1, c_1^3h^{-3k-2} = 1, c_2^3h^{-1} = 1, c_3^2h^{-1} = 1, c_4h = 1\rangle.$$

Note that $|\pi_1(K((18k+9)/(3k+1)))| = |\pi_1(K((18k+9)/(3k+2)))|$ is $24m$ for some integer $m \geq 1$ coprime to 6 [33, 43].

Note that the element h is central and generates a cyclic normal subgroup $\langle h\rangle$ in both $\pi_1(K((18k+9)/(3k+1)))$ and $\pi_1(K((18k+9)/(3k+2)))$. Let us consider the quotient groups $\pi_1(K((18k+9)/(3k+1)))/\langle h\rangle$ and $\pi_1(K((18k+9)/(3k+2)))/\langle h\rangle$, which have the same presentation:

$$\langle c_1, c_2, c_3 \mid c_1^3 = c_2^3 = c_3^2 = c_1c_2c_3 = 1\rangle.$$

This group is the tetrahedral group (spherical triangle group $\Delta(2,3,3)$) of order 12. Hence h has order $24m/12 = 2m$ in both $\pi_1(K((18k+9)/(3k+1))$ and $\pi_1(K((18k+9)/(3k+2))$.

Claim 3.2. *The slope element with slope* $(18k+9)/(3k+2)$ *does not belong to* $\langle\langle (18k+9)/(3k+1) \rangle\rangle$.

Proof. We first observe that the slope element with slope $(18k+9)/(3k+2)$ is expressed as $c_1^3 h^{-3k-2}$; see Figure 3.1. In $\pi_1(K((18k+9)/(3k+1)))$, the above presentation shows that $c_1^3 h^{3k+1} = 1$. Hence $c_1^3 h^{-3k-2} = h^{-6k-3}$. Assume for a contradiction that $c_1^3 h^{-3k-2} = h^{-6k-3} \in \langle\langle (18k+9)/(3k+1) \rangle\rangle$. Then $h^{6k+3} = 1$ in $\pi_1(K((18k+9)/(3k+1)))$. Hence $6k+3$ would be divided by $2m$. This is impossible. $\square$

Similarly we have:

Claim 3.3. *The slope element with slope* $(18k+9)/(3k+1)$ *does not belong to* $\langle\langle (18k+9)/(3k+2) \rangle\rangle$.

Proof. We first observe that the slope element with slope $(18k+9)/(3k+1)$ is expressed as $c_1^3 h^{3k+1}$; see Figure 3.1. In $\pi_1(K((18k+9)/(3k+2)))$, the above presentation shows that $c_1^3 h^{-3k-2} = 1$. Hence $c_1^3 h^{3k+1} = h^{6k+3}$. Assume for a contradiction that $c_1^3 h^{3k+1} = h^{6k+3}$ belongs to $\langle\langle (18k+9)/(3k+2) \rangle\rangle$. Then $h^{6k+3} = 1$ in $\pi_1(K((18k+9)/(3k+2)))$, and we have a contradiction. $\square$

Claims 3.2 and 3.3 shows that $\langle\langle (18k+9)/(3k+1) \rangle\rangle \neq \langle\langle (18k+9)/(3k+2) \rangle\rangle$. $\square$

Let r and r' be distinct slopes. Then Theorem 1.2 says that $\langle\langle r \rangle\rangle \neq \langle\langle r' \rangle\rangle$. However, two normal subgroups $\langle\langle r \rangle\rangle$ and $\langle\langle r' \rangle\rangle$ may be "similar" in the sense that there is an automorphism $\varphi : G(K) \to G(K)$ such that $\varphi(\langle\langle r \rangle\rangle) = \langle\langle r' \rangle\rangle$. If $\langle\langle r \rangle\rangle$ and $\langle\langle r' \rangle\rangle$ are similar, then the automorphism $\varphi : G(K) \to G(K)$ induces an isomorphism $G(K)/\langle\langle r_1 \rangle\rangle \to G(K)/\langle\langle r_2 \rangle\rangle$. For instance, if K is amphicheiral, i.e. $E(K)$ admits an orientation revering homeomorphism f, then f induces an automorphism $f_* : G(K) \to G(K)$ such that $f_*(\langle\langle r \rangle\rangle) = \langle\langle -r \rangle\rangle$, hence $\langle\langle r \rangle\rangle$ and $\langle\langle -r \rangle\rangle$ are distinct, but similar.

The next result shows that normal closures of slopes are rigid under automorphisms of $G(K)$ for prime, non-amphicheiral knots K.

Proposition 3.4. *Let* K *be a prime knot and* φ *an automorphism of* $G(K)$. *Then for any slope* $r \in \mathbb{Q}$, $\varphi(\langle\langle r \rangle\rangle) = \langle\langle r \rangle\rangle$ *or* $\langle\langle -r \rangle\rangle$. *Furthermore, if* K *is not amphicheiral, then* $\varphi(\langle\langle r \rangle\rangle) = \langle\langle r \rangle\rangle$.

Proof. When K is a prime knot, [49, Corollary 4.2] shows that any automorphism of $G(K)$ is induced by a homeomorphism of $E(K)$. Let (μ, λ) be a standard meridian-longitude pair of K. Since $\varphi(\mu) = \mu^{\varepsilon}$ $(\varepsilon = \pm 1)$ by [20] and $\varphi(\lambda) = \lambda^{\varepsilon'}$ $(\varepsilon' = \pm 1)$ for homological reasons, $\varphi(\mu^m \lambda^n) = \mu^{\varepsilon m} \lambda^{\varepsilon' n}$. This implies that $\varphi(\langle\langle m/n \rangle\rangle) = \langle\langle m/n \rangle\rangle$ or $\langle\langle -m/n \rangle\rangle$. If K is non-amphicheiral, then the homeomorphism preserves the orientation of S^3, and hence $\varepsilon = \varepsilon'$. Thus $\varphi(\langle\langle m/n \rangle\rangle) = \langle\langle m/n \rangle\rangle$. $\square$

Theorem 1.2, together with Proposition 3.4, implies the following stronger version of the peripheral Magnus property for prime, non-amphicheiral knots.

Theorem 3.5. *Let K be a nontrivial, prime knot. If $\varphi(\langle\langle r \rangle\rangle) = \langle\langle r' \rangle\rangle$ for some automorphism φ of $G(K)$, then $r' = \pm r$. Moreover, if K is not amphicheiral or $\varphi = id$, then $r' = r$.*

Proof. Following Proposition 3.4 $\langle\langle r' \rangle\rangle = \varphi(\langle\langle r \rangle\rangle)$ is either $\langle\langle r \rangle\rangle$ or $\langle\langle -r \rangle\rangle$; the latter can happen only when K is amphicheiral. Then the peripheral Magnus property (Theorem 1.2) shows $r' = r$ or $-r$, respectively. $\square$

As we have mentioned, for the torus knot $T_{3,2}$ and slopes $r = (18k+9)/(3k+1)$ and $r' = (18k+9)/(3k+2)$, $T_{3,2}(r)$ is (orientation reversingly) homeomorphic to $T_{3,2}(r')$ [36]. Thus the quotients $G(T_{3,2})/\langle\langle r \rangle\rangle$ and $G(T_{3,2})/\langle\langle r' \rangle\rangle$ are isomorphic. On the other hand, Theorem 3.5 says that no isomorphism between $G(T_{3,2})/\langle\langle r \rangle\rangle$ and $G(T_{3,2})/\langle\langle r' \rangle\rangle$ can be induced by an automorphism of $G(T_{3,2})$. This illustrates a possibility for non-similar normal subgroups yielding the same quotient groups.

We close this section by giving the following observation for conjugacy among slope elements. A knot K is *cabled* if it is a (p,q)-cable of a knot K', and we call the slope pq the *cabling slope* of a cabled knot K. In the following, we regard a torus knot as a cabled knot, the (p,q)-cable of the unknot. If K is neither a cabled knot nor a composite knot, then the peripheral subgroup $P(K)$ is malnormal [21], and hence if $g\gamma_1 g^{-1} = \gamma_2$, then g belongs to $P(K)$ and $\gamma_1 = \gamma_2$. Even when K is a cabled knot or a composite knot we have:

Proposition 3.6 (conjugation of slope elements)**.** *Let K be a nontrivial knot, and let γ_1 and γ_2 be slope elements in $P(K)$. Assume that $g\gamma_1 g^{-1} = \gamma_2$ in $G(K)$. Then $\gamma_2 = \gamma_1$. Furthermore, if $g \notin P(K)$, then K is a cabled knot or a composite knot and γ_1 represents the cabling slope or the meridional slope, respectively.*

Proof. By assumption $\langle\langle \gamma_2 \rangle\rangle = \langle\langle g\gamma_1 g^{-1} \rangle\rangle$, which coincides with $\langle\langle \gamma_1 \rangle\rangle$. Then the peripheral Magnus property (Theorem 1.2) says $\gamma_2 = \gamma_1$ or γ_1^{-1}. If $g \in P(K)$, then g commutes with γ_1 hence $\gamma_2 = g\gamma_1 g^{-1} = \gamma_1$. So we assume $g \notin P(K)$. Then we have a non-degenerate map $f : S^1 \times [0,1] \to E(K)$ such that $f(S^1 \times \{0\}) = c$ representing γ_1 and $f(S^1 \times \{1\}) = c'$ representing γ_2. Since $\gamma_2 = \gamma_1$ or γ_1^{-1}, we may assume $c \cap c' = \emptyset$. By [28, VIII.13.Annulus Theorem] we have a non-degenerate embedding $g : S^1 \times [0,1] \to E(K)$ with $g|_{S^1 \times \{0,1\}} = f|_{S^1 \times \{0,1\}}$. Consider the torus decomposition [29, 30] of $E(K)$ and denote the outermost piece which contains $\partial E(K)$ by X. The existence of an essential annulus $A = g(S^1 \times [0,1])$ in $E(K)$ implies that X is Seifert fibered. Hence K is a cabled or a composite knot. Moreover, we may further isotope A so that $A \subset X$ and it is vertical (i.e. consisting of fibers of Seifert fibration). Thus ∂A is a regular fiber. Hence γ_1 represents either a cabling slope (if K is a cabled knot), or a meridional slope (if K is a composite knot). Finally, assume for a contradiction that $\gamma_2 = \gamma_1^{-1}$. Then after abelianization the equation $g\gamma_1 g^{-1} = \gamma_1^{-1}$ implies that γ_1 is trivial in $H_1(E(K))$, i.e. γ_1 is a preferred longitude. However, a cabling slope and a meridional slope are not preferred longitude, a contradiction. So $\gamma_2 = \gamma_1$. □

4. Finitely generated normal closures of slopes

In this section we prove Theorem 1.3. The next proposition gives a classification of finitely generated, normal subgroups of infinite index of knot groups.

Proposition 4.1. *Let N be a finitely generated, nontrivial, normal subgroup of infinite index of $G(K)$. Then either*

(i) *$E(K)$ is Seifert fibered (i.e. K is a torus knot) and N is a subgroup of Seifert fiber subgroup, the subgroup generated by a regular fiber of a Seifert fibration (i.e. N is a subgroup of the center of the torus knot group G) or,*

(ii) *$E(K)$ fibers over S^1 with surface fiber Σ and N is a subgroup of finite index of $\pi_1(\Sigma)$.*

Proof. This essentially follows from the classification of finitely generated, normal subgroups of infinite index of 3–manifold groups [23], [1, p.118 (L9)]. Suppose for a contradiction that N is neither (i) nor (ii). Then it follows from [23], [1, p.118 (L9)] that $E(K)$ is the union of two twisted I-bundle over a compact connected (possibly non-orientable) surface Σ and N is a subgroup of finite index of $\pi_1(\Sigma)$. Then $G(K)$ is written as an extension

$$\{1\} \to \pi_1(\Sigma) \to G(K) \to \mathbb{Z}_2 * \mathbb{Z}_2 \to \{1\}.$$

However, this would imply $G(K)/[G(K), G(K)] \cong \mathbb{Z}$ has an epimorphism to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, the abelianization of $\mathbb{Z}_2 * \mathbb{Z}_2$, which is impossible. □

Proof of Theorem 1.3. Let us assume r is a finite surgery slope of K. Then $\pi_1(K(r)) = G(K)/\langle\langle r\rangle\rangle$ is finite. Hence $\langle\langle r\rangle\rangle$ is a subgroup of finite index of the finitely generated group $G(K)$, so it is finitely generated [35, Corollary 2.7.1].

If K is a torus knot $T_{p,q}$ and r is a cabling slope pq, then r is represented by a regular fiber t in the Seifert fiber space $E(T_{p,q})$, and $\langle\langle pq\rangle\rangle$ is the infinite cyclic normal subgroup generated by t. This means that $\langle\langle pq\rangle\rangle \cong \mathbb{Z}$.

To prove the converse, we suppose that $\langle\langle r\rangle\rangle$ is finitely generated. We divide into two cases depending upon $\langle\langle r\rangle\rangle$ has finite index in $G(K)$ or not. If it has finite index, then $G(K)/\langle\langle r\rangle\rangle = \pi_1(K(r))$ is a finite group, and hence r is a finite surgery slope.

If $\langle\langle r\rangle\rangle$ has infinite index in $G(K)$, then we have two possibilities described in Proposition 4.1. If we have the case (i) in Proposition 4.1, then K is a torus knot and $r = pq$. Now suppose for a contradiction that the case (ii) in Proposition 4.1 occurs. Then $\langle\langle r\rangle\rangle$ is a subgroup of finite index of the normal subgroup $\pi_1(\Sigma)(\subset G(K))$, in particular, r lies in the Seifert surface subgroup $\pi_1(\Sigma)$. This implies r is a preferred longitude, i.e. $r = 0$. Since $\langle\langle 0\rangle\rangle = \langle\langle \partial\Sigma\rangle\rangle \subset \pi_1(\Sigma)$ is a normal subgroup of $G(K)$, it is also normal in $\pi_1(\Sigma)$. Hence $\langle\langle \partial\Sigma\rangle\rangle$ is a normal subgroup of finite index of $\pi_1(\Sigma)$, and hence $\pi_1(\Sigma)/\langle\langle \partial\Sigma\rangle\rangle \cong \pi_1(\widehat{\Sigma})$ would be a finite group, where $\widehat{\Sigma}$ is a closed orientable surface of genus $g(K) \geq 1$ obtained by capping off Σ along $\partial\Sigma$. This is a contradiction. □

We close this section with the following.

Proposition 4.2. *Let K be a nontrivial knot and r a slope of K which is neither a finite surgery slope nor a cabling slope pq if K is a torus knot $T_{p,q}$. Then for any normal subgroup N of $G(K)$ the intersection $N \cap \langle\langle r\rangle\rangle$ is either trivial or not finitely generated.*

Proof. Theorem 1.3, together with the assumption, shows that $\langle\langle r\rangle\rangle$ is not finitely generated. Let M be the covering space of $E(K)$ associated to $\langle\langle r\rangle\rangle \subset G(K)$. Then $\langle\langle r\rangle\rangle$ is an infinitely generated 3–manifold group $\pi_1(M)$. Let us write $H = N \cap \langle\langle r\rangle\rangle$, which is normal in $G(K)$, and hence normal in $\langle\langle r\rangle\rangle$. Assume for a contradiction that H is nontrivial and finitely generated. Then [46, Theorem 3.2] shows that H is infinite cyclic. This implies that K is a torus knot $T_{p,q}$ [19] and H is contained in the infinite cyclic normal subgroup $\langle\langle pq\rangle\rangle$ generated by a regular fiber of a Seifert fibration of $E(K) = E(T_{p,q})$ [29, II.4.8.Lemma]. Hence $H \subset \langle\langle r\rangle\rangle \cap \langle\langle pq\rangle\rangle$. However, $\langle\langle r\rangle\rangle \cap \langle\langle pq\rangle\rangle$ would be trivial as we will prove in Proposition 5.4(ii). This is a contradiction. Thus $N \cap \langle\langle r\rangle\rangle$ is not finitely generated. □

5. Finite family of normal closures of slopes and their intersection

The goal of this section is to establish Theorem 1.4.

For an element $h \in G(K)$, we denote its centralizer $\{g \in G(K) \mid gh = hg\}$ by $Z(h)$. We call $h \in G(K)$ a *central element* of $G(K)$ if $Z(h) = G(K)$, and denote the *center* of $G(K)$, the normal subgroup consisting of all the central elements, by $Z(K)$.

Let r be a slope of K. Then, throughout this section, we use r to denote also a slope element $\gamma \in P(K)$ representing r. So $Z(r)$ means $Z(\gamma)$. (Note that γ^{-1} also represents r and $Z(\gamma) = Z(\gamma^{-1})$.)

Recall that $Z(r) = G(K)$ happens for some slope r if and only if K is a torus knot $T_{p,q}$ and $r = pq$; see [6] and [1, Theorem 2.5.2]. Also, we remark that by

Proposition 3.6 $Z(r) = P(K)$ unless K is cabled and r is the cabling slope, or K is composite and r is the meridional slope (we do not use this fact, though).

If $K = T_{p,q}$ and $r = pq$, then given non-central element $g \in G(K)$, obviously $aga^{-1} \in Z(r)$ for any $a \in G(K)$. Except this very restricted situation, we have:

Lemma 5.1. *Let K be a nontrivial knot and r a nontrivial slope of K. If K is a torus knot $T_{p,q}$, we assume $r \neq pq$. Then for every non-central element $g \in G(K)$, we can take an element $a \in G(K)$ so that $aga^{-1} \notin Z(r)$.*

Proof. If $g \notin Z(r)$, then take $a = 1$ to obtain the desired conclusion. So in the following we assume $g \in Z(r)$. By a structure theorem of the centralizer of 3-manifold groups [1, Theorem 2.5.1], either $Z(r)$ is an abelian group of rank at most two, or $Z(r)$ is conjugate to a subgroup of the fundamental group of a Seifert fibered piece of $E(K)$ with respect to the torus decomposition of $E(K)$ [29, 30].

First assume that $Z(r)$ is an abelian group of rank at most two. (In fact, $Z(r) = \mathbb{Z} \oplus \mathbb{Z}$, because $Z(r) \supset P(K)$.) Assume, to the contrary that $aga^{-1} \in Z(r)$ for all $a \in G(K)$. Then the normal closure $N = \langle\langle g \rangle\rangle$ of g in $G(K)$ is a nontrivial normal subgroup of $G(K)$ contained in $Z(r)$. Thus N is finitely generated. If N has finite index in $G(K)$ then $G(K)$ is virtually abelian, which cannot happen for nontrivial knot groups since the knot group contains a free group of rank ≥ 2, the fundamental group of the minimum genus Seifert surface. So N is an abelian normal subgroup of infinite index of $G(K)$. By Proposition 4.1, either K is a torus knot and N is a subgroup of the center of $G(K)$, or $E(K)$ fibers over S^1 with torus fiber. In the former case, $g \in N$ is a central element, contradicting the choice of g. In the latter case $\partial E(K) = \emptyset$ so this cannot happen, either.

Next we assume that $Z(r)$ is not an abelian group of rank at most two. Then K is a torus knot, or a satellite knot which has a Seifert fibered piece with respect to its torus decomposition. Assume first that K is a torus knot $T_{p,q}$. Then $Z(r)$ is $\mathbb{Z} \oplus \mathbb{Z}$ since $r \neq pq$ (cf. [1, Theorem 2.5.2]). This contradicts the assumption. Thus K is a satellite knot whose exterior has a Seifert fibered piece M with respect to its torus decomposition. Let T be an essential torus which is a member of the family of tori giving the torus decomposition of $E(K)$. Since T is separating, $E(K)$ is expressed as $E_1 \cup_T E_2$; we may assume $M \subset E_1$. Then $G(K)$ is a nontrivial amalgamated product $G(K) = G_1 *_H G_2$, where H denotes the fundamental group of an essential torus T and $G_i = \pi(E_i)$. The centralizer of r is a conjugate to a subgroup of $\pi_1(M) \subset G_1$, thus there is an element $u \in G(K)$ such that $uZ(r)u^{-1} \subset G_1$. Since $g \in Z(r)$, this implies $ugu^{-1} \in uZ(r)u^{-1} \subset G_1$. Let us write $w = ugu^{-1} \in G_1$.

Claim 5.2. *There exists an element $v \in G(K)$ such that $vwv^{-1} \notin G_1$.*

Proof. We divide the argument into the following two cases.

Case 1. There exists $f \in G_1$ such that $fwf^{-1} \in G_1 - H$.

We choose v' so that $v' \in G_2 - H$ and let $v = v'f$. Then by [35, Corollary 4.4.2], the canonical form of $vwv^{-1} = v'(fwf^{-1})v'^{-1}$ has length three, which is independent of a choice of right coset representatives. Hence, $vwv^{-1} \notin G_1$.

Case 2. For every $f \in G_1$, $fwf^{-1} \in H$.

Let N_1 be the normal closure $\langle\langle w \rangle\rangle_{G_1}$ of w in G_1. Then $N_1 \subset H \cong \mathbb{Z} \oplus \mathbb{Z}$ is an abelian normal subgroup of G_1. Assume that $N_1 \cong \mathbb{Z} \oplus \mathbb{Z}$ and it has finite index in G_1. Then [22, Theorem 10.6] shows that E_1 is either $S^1 \times S^1 \times [0,1]$ or a twisted I–bundle over the Klein bottle. Either case cannot occur. So N_1 has infinite index in G_1. Then referring [23] or [1, p.118 (L9)], E_1 would be the union of two twisted I–bundle over the Klein bottle, or a torus bundle over S^1. They are closed, a contradiction.

Thus $N_1 \cong \mathbb{Z}$. By [1, p.118 (L9)], E_1 is Seifert fibered and N_1 is a Seifert fiber subgroup, a subgroup generated by a regular fiber of a Seifert fibration of E_1. Then N_1 is a central subgroup of G_1, and hence $w \in \langle\langle w \rangle\rangle_{G_1} = N_1$ is central and $fwf^{-1} = w$ for all $f \in G_1$. This means that $N_1 = \langle w \rangle$.

Now we show that there is an element $v \in G_2 \subset G(K)$ such that $vwv^{-1} \notin G_1$. Assume to the contrary that for any element $v \in G_2$, $vwv^{-1} \in G_1$. Since $v \in G_2$ and $w \in H \subset G_2$, $vwv^{-1} \in G_2$. In particular, $vwv^{-1} \in G_1 \cap G_2 = H$. Let us consider the normal closure $N_2 = \langle\langle w \rangle\rangle_{G_2}$ of w in G_2. Apply the above argument to see that $vwv^{-1} = w$ for all $v \in G_2$ and $N_2 = \langle w \rangle$.

This shows that $N_1 = N_2 = \langle w \rangle$. Since w is central in both G_1 and G_2, w is a central element in the entire group $G(K)$. However, this implies $g = u^{-1}wu = w$ is a central element, contradicting the choice of g. This completes a proof of Claim 5.2 □

For $a = u^{-1}vu$, we have:

$$aga^{-1} = (u^{-1}vu)g(u^{-1}vu)^{-1} = u^{-1}(vugu^{-1}v^{-1})u = u^{-1}(vwv^{-1})u \notin u^{-1}G_1u.$$

Since $Z(r) \subset u^{-1}G_1u$, we have $aga^{-1} \notin Z(r)$, as desired. □

Now we are ready to prove:

Theorem 1.4. Let K be a nontrivial knot in S^3, and let $\{r_1, \dots, r_n\}$ $(n \geq 2)$ be any finite family of slopes of K. If K is a torus knot $T_{p,q}$, we assume that

$pq \notin \{r_1, \ldots, r_n\}$. Then $\langle\langle r_1 \rangle\rangle \cap \cdots \cap \langle\langle r_n \rangle\rangle$ is nontrivial. Moreover, this subgroup is finitely generated if and only if all the r_i are finite surgery slopes.

Proof. When r_i is the trivial slope ∞ for some $1 \le i \le n$, then $\langle\langle r_i \rangle\rangle = G(K)$, so we assume that $r_i \neq \infty$ for all $1 \le i \le n$. We first show the nontriviality of the intersection. Let

$$\Gamma_0(G(K)) \supset \Gamma_1(G(K)) \supset \cdots$$

be the lower central series of $G(K)$, defined inductively by

$$\Gamma_0(G(K)) = G(K) \quad \text{and} \quad \Gamma_i(G(K)) = [G(K), \Gamma_{i-1}(G(K))] \ (i > 0).$$

Although it is known that $\Gamma_i(G(K)) = \Gamma_1(G(K)) = [G(K), G(K)]$ for $i \ge 1$ [39, p.59], we use $\Gamma_i(G(K))$ for convenience in the following inductive construction of an element $g_m \in \Gamma_m(G(K))$ which belongs to $\bigcap_{i=1}^{m} \langle\langle r_i \rangle\rangle$ for $m = 1, \ldots, n$.

For the case $m = 1$, take $a_1 \in G(K)$ so that $a_1 \notin Z(r_1)$ (by the assumption, $Z(r_1) \neq G(K)$ so such a_1 exists). Then $g_1 = [r_1,\ a_1] = r_1(a_1 r_1^{-1} a_1^{-1})$ is a nontrivial element in $\Gamma_1(G(K)) = [G(K), G(K)]$ that belongs to $\langle\langle r_1 \rangle\rangle$ as desired.

Let us assume $m \ge 2$, and we have already found a nontrivial element g_{m-1} in $\Gamma_{m-1}(G(K))$ such that $g_{m-1} \in \bigcap_{i=1}^{m-1} \langle\langle r_i \rangle\rangle$.

Claim 5.3. *g_{m-1} is not a central element in $G(K)$.*

Proof. Assume for a contradiction that g_{m-1} is a central element in $G(K)$. Then K is a torus knot $T_{p,q}$ and $Z(K)$ is generated by a slope element pq, which is represented by regular fiber t of the Seifert fibration of $E(T_{p,q})$; see [6] and [1, Theorem 2.5.2]. Thus $Z(K) = \langle pq \rangle$ and $g_{m-1} = t^x$ for some non-zero integer x. Since $g_{m-1} \in \Gamma_{m-1}(G(K)) \subset \Gamma_1(G(K)) = [G(K), G(K)]$, it should be trivial in $G(K)/[G(K), G(K)] = H_1(E(K)) \cong \mathbb{Z}$. On the other hand, $t^x = (\mu^{pq}\lambda)^x$ becomes $\mu^{pqx} \in H_1(E(K))$, which is nontrivial, a contradiction. □

Thus by Lemma 5.1 there is $a_m \in G(K)$ such that $a_m g_{m-1} a_m^{-1} \notin Z(r_m)$. Let us take

$$g_m = [r_m,\ a_m g_{m-1} a_m^{-1}] \in [G(K), \Gamma_{m-1}(G(K))] = \Gamma_m(G(K)).$$

Since $a_m g_{m-1} a_m^{-1} \notin Z(r_m)$, $g_m \neq 1$. Obviously $g_m \in \langle\langle r_m \rangle\rangle$. Since $g_{m-1} \in \bigcap_{i=1}^{m-1} \langle\langle r_i \rangle\rangle$ and $\bigcap_{i=1}^{m-1} \langle\langle r_i \rangle\rangle$ is normal in $G(K)$, $a_m g_{m-1} a_m^{-1} \in \bigcap_{i=1}^{m-1} \langle\langle r_i \rangle\rangle$ as well. Therefore $1 \neq g_m \in \bigcap_{i=1}^{m} \langle\langle r_i \rangle\rangle$.

Next we determine when $\langle\langle r_1 \rangle\rangle \cap \cdots \cap \langle\langle r_n \rangle\rangle$ is finitely generated. Assume that at least one of $r_i \in \{r_1, \ldots, r_n\}$ is not a finite surgery slope. Without loss of generality, we may assume r_1 is not a finite surgery slope. By the assumption $r_1 \neq pq$ neither when $K = T_{p,q}$. Note that the above argument shows that $N = \bigcap_{i=1}^{n} \langle\langle r_i \rangle\rangle$ is nontrivial. Since r_1 is neither the cabling slope nor a finite surgery slope, Proposition 4.2 shows that $N = N \cap \langle\langle r_1 \rangle\rangle$ is not finitely generated.

Conversely, assume that every r_i is a finite surgery slope. Then all $\langle\langle r_i \rangle\rangle$ are subgroup of finite index of $G(K)$, so is their intersections $\bigcap_{i=1}^{n} \langle\langle r_i \rangle\rangle$. Since $G(K)$ is finitely generated, so is its subgroup $\bigcap_{i=1}^{n} \langle\langle r_i \rangle\rangle$ of finite index. □

In Theorem 1.4, if K is a torus knot, the assumption that $pq \notin \{r_1, \dots, r_n\}$ is essential. Actually, if $pq \in \{r_1, \dots, r_n\}$, then $\langle\langle r_1 \rangle\rangle \cap \cdots \cap \langle\langle r_n \rangle\rangle$ is generically trivial.

Proposition 5.4. *Let K be a torus knot $T_{p,q}$ and r a slope distinct from pq.*

(i) *If r is a finite surgery slope, then $\langle\langle pq \rangle\rangle \cap \langle\langle r \rangle\rangle \cong \mathbb{Z}$.*

(ii) *If r is not a finite surgery slope, then $\langle\langle pq \rangle\rangle \cap \langle\langle r \rangle\rangle = \{1\}$.*

Proof. Recall that the slope pq is represented by a regular fiber t of the Seifert fiber space $E(T_{p,q})$, and hence the infinite cyclic normal subgroup generated by t coincides with $\langle\langle pq \rangle\rangle$. Hence $\langle\langle pq \rangle\rangle \cap \langle\langle r \rangle\rangle = \langle t^k \rangle \subset \langle t \rangle \cong \mathbb{Z}$ for some integer $k \geq 0$. Hence, $\langle\langle pq \rangle\rangle \cap \langle\langle r \rangle\rangle$ is either trivial or infinite cyclic.

(i) If r is a finite surgery slope, t has a finite order in the finite group $G(K)/\langle\langle r \rangle\rangle$. Thus there exists an integer $k' \geq 1$ such that $t^{k'} \in \langle\langle r \rangle\rangle$ in $G(K)$. Thus $\langle\langle pq \rangle\rangle \cap \langle\langle r \rangle\rangle$ is nontrivial, and hence it is infinite cyclic.

(ii) Recall that $G(K) = \langle a, b \mid a^p = b^q \rangle$ and $t = a^p = b^q$. Thus $t^k = (a^p)^k = a^{pk} \in \langle\langle r \rangle\rangle$. This means that $a^{pk} = 1$ in $\pi_1(K(r))$. Since r is neither a reducing surgery slope nor a finite surgery slope, $\pi_1(K(r))$ has no torsion element. Therefore $a = 1$ in $\pi_1(K(r))$, unless $k = 0$. However, this implies that $G(K)$ is finite cyclic, a contradiction. Thus $k = 0$, so $\langle\langle pq \rangle\rangle \cap \langle\langle r \rangle\rangle = \{1\}$. □

Corollary 5.5. *Let K be a hyperbolic knot in S^3 which is not the $(-2, 3, 7)$–pretzel knot. Then for any finite family of slopes $r_1, \dots, r_n \in \mathbb{Q}$ with $n > 2$, the subgroup $\cap_{i=1}^{n} \langle\langle r_i \rangle\rangle$ is not finitely generated.*

Proof. Since K is a hyperbolic knot, [42, Theorem 1.4] shows that it has at most three nontrivial finite surgeries, and except when K is the $(-2, 3, 7)$–pretzel knot, it has at most two such surgeries. Thus the result follows from Theorem 1.4 immediately. □

Example 5.6. Let K be the $(-2, 3, 7)$–pretzel knot. Then it has three nontrivial finite surgery slopes: $17, 18$ and 19. Hence, $\langle\langle 17 \rangle\rangle \cap \langle\langle 18 \rangle\rangle \cap \langle\langle 19 \rangle\rangle$ is a finitely generated nontrivial normal subgroup of $G(K)$.

6. Chains of normal closures of slopes

Applying Propositions 2.3 and 2.4 we have:

Theorem 1.10. Let K be a non-torus knot in S^3. If $\langle\langle r_1 \rangle\rangle \supset \cdots \supset \langle\langle r_n \rangle\rangle$ for mutually distinct slopes $r_1, \dots, r_n \in \mathbb{Q}$, then $n \leq 2$. In particular, there is no infinite descending chain nor ascending chain of normal closures of slopes.

Proof. Assume for a contradiction that we have $\langle\langle r_1 \rangle\rangle \supset \langle\langle r_2 \rangle\rangle \supset \langle\langle r_3 \rangle\rangle$ for mutually distinct slopes r_1, r_2 and r_3. Then by Proposition 2.3, r_1 and r_2 are both finite surgery slopes. However, this contradicts Proposition 2.4. □

As shown in Proposition 2.3, the inclusion $\langle\langle r \rangle\rangle \supset \langle\langle r' \rangle\rangle$ can occur only when r is a finite surgery slope. In fact, for a given finite surgery slope r, we can find infinitely many slopes r_k so that $\langle\langle r \rangle\rangle \supset \langle\langle r_k \rangle\rangle$.

Proposition 6.1. *Let K be a nontrivial knot with finite surgery slope $m/n \in \mathbb{Q}$. Let f be the order of the meridian μ in the finite group $G(K)/\langle\langle m/n \rangle\rangle$. Then*

$$\langle\langle m/n \rangle\rangle \supset \langle\langle m/kn \rangle\rangle$$

for any non-zero integer k such that $\gcd(k, m) = 1$ and $k \equiv 1 \pmod f$. If m/n is a cyclic surgery slope, then the last condition $k \equiv 1 \pmod f$ is redundant.

Proof. In $G(K)/\langle\langle m/n \rangle\rangle$, we have $\mu^m \lambda^n = 1$, hence $\lambda^n = \mu^{-m}$. Then the slope element $\mu^m \lambda^{kn}$ is equal to $\mu^{m(1-k)}$ there. If m/n is a cyclic surgery slope, then $\mu^m = 1$, since $G(K)/\langle\langle m/n \rangle\rangle = \mathbb{Z}_{|m|}$. Otherwise, $\mu^{k-1} = 1$, since $k \equiv 1 \pmod f$ and $\mu^f = 1$ in $G(K)/\langle\langle m/n \rangle\rangle$. Thus $\mu^m \lambda^{kn} = \mu^{m(1-k)} = 1$ in $G(K)/\langle\langle m/n \rangle\rangle$, i.e. $\mu^m \lambda^{kn} \in \langle\langle m/n \rangle\rangle$. Hence $\langle\langle m/kn \rangle\rangle \subset \langle\langle m/n \rangle\rangle$. □

Example 6.2. Among hyperbolic Montesinos knots, the $(-2, 3, 7)$-pretzel knot and $(-2, 3, 9)$-pretzel knot are the only ones that admit nontrivial finite surgeries [25]. To determine the order of the meridian in the resulting finite group, we used presentations given in [38] and `GAP` (Groups, Algorithms, Programming) [13].

(1) Let K be the $(-2, 3, 7)$-pretzel knot. Then 18 and 19 are cyclic surgery slopes. Hence, by Proposition 6.1, for any non-zero integer k, k' such that $\gcd(k, 18) = 1$ and $\gcd(k', 19) = 1$,

$$\langle\langle 18 \rangle\rangle \supset \langle\langle 18/k \rangle\rangle, \quad \langle\langle 19 \rangle\rangle \supset \langle\langle 19/k' \rangle\rangle.$$

Also, we may observe that the slope 17 is a finite, non-cyclic surgery slope using Montesinos trick [37]. Precisely, $K(17)$ is a Seifert fibered manifold with Seifert invariant $S^2(0; 1/3, -2/5, -1/2)$; see also [9], [10, Proposition 5.6] and [7, 4.2] for some corrections of mistakes in Proposition 5.6 in [10]. Hence its fundamental group is $I_{120} \times \mathbb{Z}_{17}$, where I_{120} is the binary icosahedral group of order 120. In the finite group $G(K)/\langle\langle 17 \rangle\rangle$ of order 2040, the order of the meridian is 170. Hence for any non-zero integer k such that $\gcd(k, 17) = 1$ and $k \equiv 1 \pmod{170}$,

$$\langle\langle 17 \rangle\rangle \supset \langle\langle 17/k \rangle\rangle.$$

(2) Let K be the $(-2, 3, 9)$-pretzel knot. It has two non-cyclic finite surgery slopes 22 and 23. Seifert invariants of Seifert fibered manifolds obtained by 22 and 23 surgeries on K were given by [2], but it contains miscalculations, so we give their corrections below. Using Montesinos trick, we observe that $K(22)$ is

a Seifert fibered manifold with Seifert invariant $S^2(0; 1/2, -1/4, 2/3)$. Its fundamental group is $O_{48} \times \mathbb{Z}_{11}$, where O_{48} is the binary octahedral group of order 48. In the finite group $G(K)/\langle\langle 22 \rangle\rangle$ of order 528, the order of the meridian is 44. Similarly, we observe that $K(23)$ is a Seifert fibered manifold with Seifert invariant $S^2(0; 1/2, 2/3, -2/5)$, whose fundamental group is $I_{120} \times \mathbb{Z}_{23}$ of order 2760. The order of the meridian is 138 in this group. Hence for any non-zero integers k, k' such that $\gcd(k, 528) = 1$ and $k \equiv 1 \pmod{44}$, $\gcd(k', 2760) = 1$ and $k' \equiv 1 \pmod{138}$,

$$\langle\langle 22 \rangle\rangle \supset \langle\langle 22/k \rangle\rangle, \quad \langle\langle 23 \rangle\rangle \supset \langle\langle 23/k' \rangle\rangle.$$

In the rest of this section, we focus on normal closures of slopes for torus knots.

Theorem 6.3. *Let K be a torus knot $T_{p,q}$. Then there is no infinite ascending chain $\langle\langle r \rangle\rangle \subset \langle\langle r_1 \rangle\rangle \subset \langle\langle r_2 \rangle\rangle \subset \cdots$.*

Proof. Suppose for a contradiction that there is an infinite ascending chain

$$\langle\langle r \rangle\rangle \subset \langle\langle r_1 \rangle\rangle \subset \langle\langle r_2 \rangle\rangle \subset \cdots .$$

By Proposition 2.3, all r_i but r are finite surgery slopes. Let $r_i = m_i/n_i$ for $i \geq 1$. We may assume that $m_i > 0$. Since $\langle\langle r_i \rangle\rangle \subset \langle\langle r_{i+1} \rangle\rangle$, we have a canonical epimorphism

$$G(K)/\langle\langle r_i \rangle\rangle \to G(K)/\langle\langle r_{i+1} \rangle\rangle.$$

This induces an epimorphism

$$H_1(K(m_i/n_i)) \cong \mathbb{Z}_{m_i} \to H_1(K(m_{i+1}/n_{i+1})) \cong \mathbb{Z}_{m_{i+1}}.$$

Thus we have an infinite sequence $m_1 \geq m_2 \geq \cdots \geq m_i \geq m_{i+1} \geq \cdots$, where m_i is divided by m_{i+1}. Hence there is a constant $N > 0$ such that $m_{i+1} = m_i$ for $i \geq N$; we denote this constant m. As stated in Case 2 of the proof of Theorem 1.2, $|pqn_i - m| \in \{1, 2, 3, 4, 5\}$. This implies that there are only finitely many possibilities for n_i, a contradiction. □

Theorem 6.4. *Let K be a torus knot $T_{p,q}$ $(p > q \geq 2)$. For each finite surgery slope $r \in \mathbb{Q}$, there exists an infinite descending chain*

$$\langle\langle r \rangle\rangle \supset \langle\langle r_1 \rangle\rangle \supset \langle\langle r_2 \rangle\rangle \supset \cdots .$$

Proof. Assume first that r is a cyclic surgery slope. Then $r = pq + \frac{1}{n}$ for some non-zero integer n. Set $r_1 = pq + \frac{1}{(pq+1)n+1}$. We show that $\langle\langle r \rangle\rangle \supset \langle\langle r_1 \rangle\rangle$.

Using standard meridian-longitude pair $(\mu,\ \lambda)$ of K, r is represented by $\mu^{pqn+1}\lambda^n$ and r_1 is represented by $\mu^{pq((pq+1)n+1)+1}\lambda^{(pq+1)n+1}$. Since $G(K)/\langle\langle r \rangle\rangle$ is abelian, the longitude λ is trivial in $G(K)/\langle\langle r \rangle\rangle$, and hence,

$$\mu^{pqn+1}\lambda^n = \mu^{pqn+1},$$

$$\mu^{pq((pq+1)n+1)+1}\lambda^{(pq+1)n+1} = \mu^{pq((pq+1)n+1)+1} = \mu^{(pq+1)(pqn+1)}$$

in $G(K)/\langle\langle r\rangle\rangle$. Since $\mu^{pqn+1} = 1$ in $G(K)/\langle\langle r\rangle\rangle$, so is $\mu^{(pq+1)(pqn+1)} = 1$. Thus $r_1 \in \langle\langle r\rangle\rangle$. We remark that r_1 remains a cyclic surgery slope. Repeat this process to obtain a desired infinite descending chain.

Next assume that r is a non-cyclic finite surgery slope m/n. This is possible only for $(p, q, |pqn-m|) = (p, 2, 2)$, $(3, 2, 3)$, $(3, 2, 4)$, $(3, 2, 5)$, $(5, 2, 3)$, $(5, 3, 2)$. Put $d = pqn - m$, and let $(p, q, |d|)$ be one of these triples. Set $r_1 = \frac{m_1}{n_1} = \frac{m(f+1)+df}{n(f+1)}$, where f is the order of the meridian μ in the finite group $G(K)/\langle\langle r\rangle\rangle$. Then we show that $\langle\langle r\rangle\rangle \supset \langle\langle r_1\rangle\rangle$. In $G(K)/\langle\langle r\rangle\rangle$, $\mu^m\lambda^n = 1$, so $\lambda^n = \mu^{-m}$. Hence the slope element $\mu^{m_1}\lambda^{n_1} = \mu^{m(f+1)+df}\lambda^{n(f+1)}$ is equal to $\mu^{m(f+1)+df-m(f+1)} = \mu^{df} = 1$ in $G(K)/\langle\langle r\rangle\rangle$, so the slope element $\mu^{m_1}\lambda^{n_1}$ representing r_1 belongs to $\langle\langle r\rangle\rangle$. This means that $\langle\langle r_1\rangle\rangle \subset \langle\langle r\rangle\rangle$. We remark that $pqn_1 - m_1 = d = pqn - m$. Then applying the above argument to $r_2 = \frac{m_2}{n_2} = \frac{m_1(f_1+1)+df_1}{n_1(f_1+1)}$, where f_1 is the order of μ in $G(K)/\langle\langle r_1\rangle\rangle$, we see that r_2 belongs to $\langle\langle r_1\rangle\rangle$. Repeat this process to obtain an infinite descending chain. □

Theorem 1.11 follows from Theorems 6.3 and 6.4.

Proposition 6.5. *Let K be a torus knot $T_{p,q}$ $(p > q \geq 2)$.*

(i) *For the infinite descending chain*

$$\langle\langle pq + \frac{1}{n}\rangle\rangle \supset \langle\langle pq + \frac{1}{(pq+1)n+1}\rangle\rangle \supset \cdots$$

consisting of cyclic surgery slopes, the intersection of these normal closures is the commutator subgroup $[G(K), G(K)]$ of $G(K)$.

(ii) *For the set of cyclic surgery slopes $\mathcal{S}$, $\bigcap_{r\in\mathcal{S}}\langle\langle r\rangle\rangle = [G(K), G(K)]$.*

Proof. (i) For simplicity, we denote the above sequence by $N_1 \supset N_2 \supset \cdots$. Then each quotient $G(K)/N_i$ is cyclic, so $[G(K), G(K)] \subset N_i$. Hence $[G(K), G(K)] \subset \bigcap_{i=1}^{\infty} N_i$. Conversely, take an element $g \notin [G(K), G(K)]$. Then $g = \mu^k h$, where μ is a meridian, $h \in [G(K), G(K)]$ and $k \neq 0$. However, there exists j such that the order of the cyclic group $G(K)/N_j$ is bigger than $|k|$. Then the epimorphism $G(K) \to G(K)/[G(K), G(K)] \to G(K)/N_j$ sends g to a nontrivial element. This means $g \notin N_j$. Hence $[G(K), G(K)] = \bigcap_{i=1}^{\infty} N_i$.

(ii) Since $G(K)/\langle\langle r\rangle\rangle$ is cyclic, $[G(K), G(K)] \subset \langle\langle r\rangle\rangle$ for any $r \in \mathcal{S}$. Thus $[G(K), G(K)] \subset \bigcap_{r\in\mathcal{S}}\langle\langle r\rangle\rangle$. Conversely, $\bigcap_{r\in\mathcal{S}}\langle\langle r\rangle\rangle \subset \bigcap_{i=1}^{\infty} N_i = [G(K), G(K)]$. Hence $\bigcap_{r\in\mathcal{S}}\langle\langle r\rangle\rangle = [G(K), G(K)]$. □

Question 6.6. *For the infinite descending chain*

$$\langle\langle r\rangle\rangle \supset \langle\langle r_1\rangle\rangle \supset \langle\langle r_2\rangle\rangle \supset \cdots$$

consisting of non-cyclic finite surgery slopes in Theorem 6.4, what is the intersection of these normal closures?

Funding

This work was supported by Japan Society for the Promotion of Science, KAKENHI Grant Numbers [JP15K17540 and JP16H02145 to I.T., JP26400099 to K. M., JP16K05149 to M.T.]; and Joint Research Grant of Institute of Natural Sciences at Nihon University for 2017, 2018 to K. M.

Acknowledgements We would like to thank the referee for a careful reading and useful comments, and Toshio Sumi for some help of `GAP` at the beginning of this project and Ederson Dutra for useful comments.

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