Vanishing nontrivial elements in a knot group by Dehn fillings

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Abstract

Let K be a nontrivial knot in S^3 with the exterior $E(K)$, and denote $\pi_1(E(K))$ by $G(K)$. We prove that for any hyperbolic knot K and any nontrivial element $g \in G(K)$, there are only finitely many Dehn fillings of $E(K)$ which trivialize g. We also demonstrate that there are infinitely many nontrivial elements in $G(K)$ which cannot be trivialized by nontrivial Dehn fillings.

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1. Introduction

Let K be a nontrivial knot in S^3 with its exterior $E(K)$. We denote the knot group $\pi_1(E(K))$ by $G(K)$. An isotopy class of unoriented simple closed curves in $\partial E(K)$ is called a *slope* on K, which is identified with $r \in \mathbb{Q} \cup \{1/0\}$ using a standard meridian-longitude of K. Denote by $K(r)$ the 3–manifold obtained by a Dehn filling of $E(K)$ along a slope r. If $r = 1/0$, i.e. it is represented by a meridional slope, then $\pi_1(K(r)) = \{1\}$ and hence, all the elements in $G(K)$ become trivial by $1/0$–filling of $E(K)$. Conversely, the Property P [13] says that only $1/0$–Dehn filling can trivialize all the elements in $G(K)$. For a given

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$r \in \mathbb{Q}$ and a nontrivial element $g \in G(K)$, in general, it is difficult to determine whether g is trivialized by the r–Dehn filling. In this note we prove:

Theorem 1.1.

(1) *Let K be a hyperbolic knot. Then for each nontrivial element $g \in G(K)$, there are only finitely many Dehn fillings of $E(K)$ which trivialize g.*
(2) *Let K be a nontrivial torus knot. Then there are infinitely many nontrivial elements $g \in G(K)$ which can be trivialized by infinitely many Dehn fillings of $E(K)$.*

Theorem 1.1 (1) asserts that for any infinite family of slopes $\{r_1, r_2, \cdots\}$ on a hyperbolic knot, no nontrivial element can be trivialized by r_i–Dehn fillings for all $i \geq 1$. So it may be interesting to compare Theorem 1.1 (1) with the following result given in [10].

Theorem 1.2 ([10]). *Let K be a hyperbolic knot in S^3, and let $\{r_1, \dots, r_n\}$ be any finite family of slopes on K. Then there are infinitely many elements $g \in G(K)$ which can be trivialized by r_i–Dehn filling of $E(K)$ for each $1 \leq i \leq n$. Furthermore, these elements form a finitely generated subgroup if and only if all the r_i are finite surgery slopes meaning that $\pi_1(K(r_i))$ is finite.*

On the other hand, there are also nontrivial elements having a high tolerance to Dehn fillings.

Theorem 1.3. *For any nontrivial knot K, there are infinitely many elements $g \in G(K)$ which remain nontrivial after any nontrivial Dehn filling of $E(K)$.*

By the loop theorem the inclusion map $i : \partial E(K) \to E(K)$ induces a monomorphism $i_* : \pi_1(\partial E(K), *) \to \pi_1(E(K), *)$, where we choose a base point $*$ in $\partial E(K)$. We denote the *peripheral subgroup* $i_*(\pi_1(\partial E(K), *))$ by $P(K)$. A primitive element γ in $P(K) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is represented by a simple closed curve on $\partial E(K)$, is called a *slope element* in $G(K)$.

Denote by $\langle\!\langle \gamma \rangle\!\rangle$ the normal closure of γ in $G(K)$. Two slope elements γ and its inverse γ^{-1} represent the same slope which is identified with $p/q \in \mathbb{Q} \cup \{1/0\}$. Since $\langle\!\langle \gamma \rangle\!\rangle = \langle\!\langle \gamma^{-1} \rangle\!\rangle$, it is convenient to denote them by $\langle\!\langle p/q \rangle\!\rangle$. Thus each slope defines the normal subgroup $\langle\!\langle p/q \rangle\!\rangle \subset G(K)$, which will be referred to as the *normal closure of the slope p/q* for simplicity. For each slope r, r–Dehn filling gives the following short exact sequence:

$$1 \to \langle\!\langle r \rangle\!\rangle \to G(K) \to G(K)/\langle\!\langle r \rangle\!\rangle = \pi_1(K(r)) \to 1.$$

Hence, $\langle\!\langle r \rangle\!\rangle$ is the subgroup of $G(K)$ consisting of elements which vanish in $\pi_1(K(r))$.

For any nontrivial knot K in S^3, nontrivial slopes r define mutually distinct normal subgroups $\langle\!\langle r \rangle\!\rangle$; see Theorem 1.2 in [10].

In terms of normal closures of slope elements, Theorems 1.1(1), 1.2 and 1.3 can be rephrased as follows.

Corollary 1.4. (1) *Let K be a hyperbolic knot in S^3. Then for any infinite family of slopes $\{r_1, r_2, \dots\}$, we have $\bigcap_{i=1}^{\infty} \langle\langle r_i \rangle\rangle = \{1\}$.*

(2) *Let K be a hyperbolic knot and let $\{r_1, \dots, r_n\}$ be any finite family of slopes. Then $\bigcap_{i=1}^{n} \langle\langle r_i \rangle\rangle \neq \{1\}$.*

(3) *Let K be a nontrivial knot in S^3. Then $G(K) - \bigcup_{r \in \mathbb{Q}} \langle\langle r \rangle\rangle$ contains infinitely many elements.*

In Section 2 we will establish a general result, Theorem 2.1, which immediately implies Theorem 1.1 (1). Although we do not mention explicitly, Theorem 2.1 can be generalized to a complete hyperbolic 3–manifold with more than one cusp. Theorem 1.1 (2) will be shown in Section 3. In Section 4, we will prove Theorem 1.3. In Section 5 we will give some numerical results concerning Theorem 1.1 (1).

In what follows we will use int$X$ to mean the interior of X, and if $X \subset M$, then $N(X)$ will denote a tubular neighborhood and $\mathcal{N}(X)$ will denote an open tubular neighborhood of X in M, and $E(X)$ will denote the exterior $M - \mathcal{N}(X)$.

We close the introduction with the following question.

Question 1.5. *Let K be a satellite knot. Then for each nontrivial element $g \in G(K)$, are there only finitely many Dehn fillings of $E(K)$ which trivialize g?*

2. Homotopically nontrivial loops and Dehn fillings on hyperbolic 3–manifolds

The goal in this section is to prove Theorem 1.1. To this end we will prove a more general result, Theorem 2.1 below. Its proof relies on the arguments used in the proof of [1, Theorem 6.2] (cf. [14]).

Let S be a surface of finite type meaning that S has finitely many boundary components and punctures and M a hyperbolic 3–manifold with a distinguished torus-cusp. Let $f : S \to M$ be a map such that every puncture is mapped properly into a cusp, which is not necessarily an embedding or an immersion. We say that f is *incompressible* if every simple loop c in S for which $f(c)$ is homotopically trivial in M bounds a disk in S, and f is *∂–incompressible* if for any proper map $b : U = [0, \infty) \times \mathbb{R} \to M$ with $b(\partial U) \subset f(S)$, there is a map $b' : \partial U \to S$ such that $f \circ b' = f|_{\partial U}$ and $b'(\partial U)$ is a proper simple line in S which bounds a properly embedded half-plane in S. We say that f is *essential* if it is incompressible and ∂–incompressible. Note that any map $g : S \to M$ properly homotopic to an essential map f is also essential.

For a compact, orientable 3–manifold X with a single torus boundary component T. Denote by $X(r)$ the 3–manifold $X \cup_r (S^1 \times D^2)$ obtained from X by r–Dehn filling along T. The core of the filled solid torus is denoted by K_r^*, or simply by K^*.

Theorem 2.1. *Let X be a compact, orientable 3–manifold whose boundary consists of a single torus T. Assume that* $\mathrm{int}X$ *admits a complete hyperbolic structure of finite volume. Then for each homotopically essential loop $c \subset X$, there are only finitely many slopes r on T such that c becomes null-homotopic in $X(r)$.*

Proof. Divide the argument into two cases depending upon c is freely homotopic into T or not.

Case 1. We assume that c is freely homotopic into T. Then c is freely homotopic in X to an essential (possibly non-simple) loop which is a multiple of a curve representing a slope, say r_0, on T. Then, of course, c is homotopically trivial in $X(r_0)$. Suppose that c becomes homotopically trivial in $X(r)$ for some slope r other than r_0. We show that there are only finitely many such slopes r. Since $r \neq r_0$, c is freely homotopic in $X(r)$ to some multiple of K^*, the core of the attached solid torus via the r–surgery. By the assumption c is homotopically trivial in $X(r)$. This then implies that K^* represents an element with finite order in $\pi_1(X(r))$. Then the length of the slope r is less than or equal to 6 on some cusp, for otherwise, as shown in [1, the 1st paragraph of Proof of Theorem 6.2], K^* represents an element with infinite order in $\pi_1(X(r))$. Although the number of slopes with length ≤ 6 depends on the size of the torus-cusp, there are only finitely many such slopes. Thus there are only finitely many slopes r such that c becomes null-homotopic in $X(r)$.

Case 2. Suppose that c is not freely homotopic in X to an essential (possibly non-simple) loop on $T \subset \partial X$. Then since c is null-homotopic in $X(r)$, we have a map $d : D^2 \to X(r)$ with $d(\partial D^2) = c$. It follows from [1, Lemma 3.2], together with its proof, we have the following.

Lemma 2.2. *We can homotope $d : D^2 \to X(r)$ keeping $d(\partial D^2) = c$ so that $d|_{d^{-1}(X)}$ is an essential map into $X \subset X(r)$, or otherwise we can find a sub-disk $\Delta \subset D^2$ and a map $\delta : \Delta \to X(r)$ transverse to the core K^* such that $\delta(\partial\Delta) \subset \mathcal{N}(K^*)$ and $\delta|_{\delta^{-1}(X)}$ is an essential map. In the former case $d^{-1}(N(K^*)) \subset D^2$ consists of disks, and in the latter case $\delta^{-1}(N(K^*)) \subset \Delta$ consists of disks and an annular neighborhood of $\partial\Delta$.*

We first consider the latter situation.

Case 2 - (i). Assume that there exists a map $\delta : \Delta \to X(r)$ transverse to the core K^* of the filled solid torus such that $\delta(\partial\Delta) \subset \mathcal{N}(K^*)$ and $\delta|_{\delta^{-1}(X)}$ is an essential map.

As mentioned in Lemma 2.2, $\delta^{-1}(N(K^*)) \subset \Delta$ consists of $n-1$ disks and an annular neighborhood of $\partial\Delta$. We note that $X(r) - K^*$ is homeomorphic to $\mathrm{int}X$, and hence, $X(r) - K^*$ is also a cusped hyperbolic 3–manifold.

Let us consider $\delta^{-1}(X) \subset \Delta$, which is Δ with $n-1$ open disks and one open collar neighborhood of $\partial\Delta$ removed. Thus $\delta^{-1}(X)$ is regarded as a sphere with n open disks removed. Then extend $\delta^{-1}(X)$ by adding n cusps along the boundary to obtain a sphere with n punctures S. Let $f : S \to X(r) - K^*$ be an essential map obtained as a natural extension of $\delta|_{\delta^{-1}(X)} : \delta^{-1}(X) \to X$ such that $S - \delta^{-1}(\mathrm{int}X)$ can be regarded as cusps of S and theses cusps are sent properly to the cusp $N(K^*) - K^*$ of $X(r) - K^*$. Since f is essential and K^* is a hyperbolic knot in $X(r)$ (equivalently int$X$ is hyperbolic), S has at least three punctures ($n \geq 3$), and hence, $\chi(S) = 2 - n < 0$. Also, by construction, all but one punctures (corresponding to $\partial\Delta$) of S have the slope r on a torus-cusp in $X(r) - K^*$. We denote the (fixed) torus-cusp $N(K^*) - K^*$ by $\mathcal{C}$ and denote the length of the slope r in $\partial\mathcal{C}$ by $l_{\partial\mathcal{C}}(r)$. Then by [1, Theorem 5.1], we have the inequality:

$$(n-1)l_{\partial\mathcal{C}}(r) \leq 6|\chi(S)| = 6(n-2).$$

This implies that $l_{\partial\mathcal{C}}(r) \leq \dfrac{6(n-2)}{n-1} < 6$. Since there are only finitely many such slopes r, c is homotopically trivial in $X(r)$ for only finitely many slopes r on T.

Case 2 - (ii). Assume that we can homotope $d : D^2 \to X(r)$ keeping $d(\partial D^2) = c$ so that $d|_{d^{-1}(X)}$ is an essential map into $X \subset X(r)$.

As in Case 2-(i), consider $d^{-1}(X) \subset D^2$, which is D^2 with n open disks removed. Note that $S = d^{-1}(X(r) - K^*)$ is the disk D^2 with n punctures, which is a natural extension of $d^{-1}(X) = d^{-1}(X(r) - \mathcal{N}(K^*))$ by adding n punctured disks along the inner boundary components of $d^{-1}(X)$. We think of these punctured disks as cusps of S. Then $d|_S : S \to X(r) - K^*$ is an essential map which extends the essential map $d|_{d^{-1}(X)}$.

Since $d|_S(\partial S) = c$ can be homotoped to a unique closed geodesic c' in the cusped hyperbolic 3–manifold $X(r) - K^*$, we can properly homotope $d|_S : S \to X(r) - K^*$ to an essential map $f : S \to X(r) - K^*$ so that $f(\partial S)$ is the geodesic c'. Also, since $f(\partial S)$ (which is homotopic to the original $c = d(\partial D^2)$) is not freely homotopic into $T \subset \partial X$ by the assumption of Case 2, we see that the number of punctures of S is at least 2 ($n \geq 2$), and hence, $\chi(S) = 1 - n < 0$. Then, by [1, Lemma 4.1], we can find a hyperbolic metric on S and a map $g : S \to X(r) - K^*$ with $g(\partial S) = f(\partial S) = c'$ such that g is pleated, i.e. $g(\partial S)$ is a geodesic in $X(r) - K^*$ and int$S$ is piecewise made of triangles such that the image of each triangle under g is an ideal hyperbolic geodesic triangle in $X(r) - K^*$ and the 1–skeleton together with ∂S forms a lamination in S. Furthermore, $g|_{\mathrm{int}S}$ is homotopic to $f|_{\mathrm{int}S}$, and $g|_{\partial S}$ is an isometry. Note that the pleated surface S has a natural induced hyperbolic metric, where the lamination is geodesic. Since $g|_{\mathrm{int}S}$ is homotopic to $f|_{\mathrm{int}S}$, $g(\partial S)$ is homotopic to $f(\partial S)$ which is a closed geodesic in $X(r) - K^*$. Uniqueness of closed geodesic in a homotopy class shows $g(\partial S) = f(\partial S)$.

Although c lives in X, the closed geodesic c' homotopic to c may intersect $N(K^*)$. Now we take a sufficiently small torus-cusp in $X(r) - K^*$ so that $g(\partial S) = c'$ does not intersect the torus-cusp; we denote this torus-cusp by $\mathcal{C}'$. It should be noted that the torus-cusp $\mathcal{C}'$ depends only on the homotopy class of the original loop c and does not depend on the slope r. Let $\mathcal{C}_S$ be the preimage $g^{-1}(\mathcal{C}') \subset S$ which consists of n horocusps corresponding to n punctures of S. The length of each component of $\partial\mathcal{C}_S$ on S is greater than or equal to the length of the corresponding slope $l_{\partial\mathcal{C}'}(r)$ on $\partial\mathcal{C}'$. Take the double DS of S along its geodesic boundary ∂S, which can be regarded as a sphere with $2n$ punctures. Then apply the argument given in [1, Lemma 5.1] (cf. [1, p.442]) to DS, we have the following inequality:

$$2n \cdot l_{\partial\mathcal{C}'}(r) \leq 6|\chi(DS)| = 6 \cdot 2(n-1).$$

This then implies that $l_{\partial\mathcal{C}'}(r) \leq \dfrac{6(n-1)}{n} < 6$. Since we have only finitely many such slopes r, there are only finitely many slopes r such that c becomes homotopically trivial in $X(r)$.

The proof of Theorem 2.1 is completed. □

We may prove Theorem 2.1 using an argument in [3].

Proof of Theorem 1.1. Let X be the exterior $E(K)$. Then $X(r) = K(r)$. Let $\alpha : S^1 \to X$ be a map such that $c = \alpha(S^1)$ is freely homotopic to a representative of a non-trivial element $g \in \pi_1(X)$; c is homotopically essential in X. Then g becomes trivial in $\pi_1(X(r))$ if and only if c is homotopically trivial in $X(r)$. Now the result follows from Theorem 2.1. □

If X is the exterior of a hyperbolic knot K in S^3, i.e. $X = S^3 - \mathcal{N}(K)$, then we have a sharper conclusion in Case 1 in the proof of Theorem 2.1 as follows.

Proposition 2.3. *Let K be a hyperbolic knot in S^3 and c a homotopically essential loop in $E(K)$. Suppose that c is freely homotopic into $T = \partial E(K)$. Then there are at most four nontrivial slopes r for which c is null-homotopic in $K(r)$.*

Proof. Since c is freely homotopic into $T = \partial E(K)$, it is freely homotopic to c'^n for some simple loop c' on T. By the assumption c is homotopically essential, so is c' and $n \neq 0$. We may assume c' represents $\mu^p\lambda^q \in P(K)$. Suppose that $r \neq p/q$ and c is null-homotopic in $K(r)$. Then $(\mu^p\lambda^q)^n = 1$ in $\pi_1(K(r))$. This implies that $\langle\langle r\rangle\rangle \ni (\mu^p\lambda^q)^n$. It follows from Lemma 2.4 below that r is a finite surgery slope. Ni and Zhang [17] show that a hyperbolic knot admits at most three finite surgery slopes and if it has three such slopes, then K is the $(-2,3,7)$–pretzel knot. Thus, including the slope p/q, K has at most four nontrivial slopes r for which c is null-homotopic in $K(r)$. □

Lemma 2.4. *Let K be a nontrivial knot in S^3. Let γ_1 and γ_2 be nontrivial distinct slope elements in $G(K)$. Assume that $\langle\langle\gamma_1\rangle\rangle \ni \gamma_2^n$ for some non-zero integer n, then γ_1 is a finite surgery slope element.*

Proof. Write $\gamma_1 = \mu^{p_1}\lambda^{q_1}$ and $\gamma_2 = \mu^{p_2}\lambda^{q_2}$ for some relatively prime integers p_i, q_i $(i = 1, 2)$. Since $(\mu^{p_2}\lambda^{q_2})^n = \mu^{p_2 n}\lambda^{q_2 n} \in \langle\langle\mu^{p_1}\lambda^{q_1}\rangle\rangle$,

$$(\mu^{p_1}\lambda^{q_1})^{q_2 n}(\mu^{p_2 n}\lambda^{q_2 n})^{-q_1} = \mu^{(p_1 q_2 - q_1 p_2)n} \in \langle\langle\mu^{p_1}\lambda^{q_1}\rangle\rangle.$$

Since $p_1/q_1 \neq p_2/q_2$ and $n \neq 0$, $(p_1q_2 - q_1p_2)n \neq 0$. Thus [10, Lemma 2.1] shows that $r_1 = p_1/q_1$ is a finite surgery slope or a reducing surgery slope, and [10, Lemma 2.2] excludes the latter possibility. $\square$

Lemma 2.4 is a slight generalization of Proposition 2.3 in [10].

3. The commutator subgroups of torus knot groups and Dehn fillings

For any hyperbolic knot, Theorem 1.1 (1) shows that only finitely many $\langle\langle r_i \rangle\rangle$ can intersect nontrivially. On the other hand, this is not the case for nontrivial torus knots.

We first observe the following elementary fact.

Lemma 3.1. *Let r be a slope of a nontrivial knot K. Then r is a cyclic surgery slope (i.e. $\pi_1(K(r))$ is cyclic) if and only if $[G(K), G(K)] \subset \langle\langle r \rangle\rangle$.*

Proof. Since $\pi_1(K(r)) = G(K)/\langle\langle r \rangle\rangle$ is abelian, $\langle\langle r \rangle\rangle$ contains the commutator subgroup $[G(K), G(K)]$. Conversely, if $[G(K), G(K)] \subset \langle\langle r \rangle\rangle$, then $\pi_1(K(r)) = G(K)/\langle\langle r \rangle\rangle$ is abelian. Hence $\pi_1(K(r)) \cong H_1(K(r))$, which is cyclic. $\square$

Proof of Theorem 1.1 (2). Let K be a torus knot $T_{p,q}$ $(p > q \geq 2)$. Then $(pqn \pm 1)/n$–surgery results in a lens space [16]. Hence by Lemma 3.1, for a torus knot $T_{p,q}$, $\cap_{n\in\mathbb{Z}} \langle\langle (pqn \pm 1)/n \rangle\rangle \supset [\pi_1(E(T_{p,q})), \pi_1(E(T_{p,q}))]$. In particular, a longitude becomes a trivial element after infinitely many Dehn fillings. Note that since a torus knot is fibered, $[\pi_1(E(T_{p,q})), \pi_1(E(T_{p,q}))]$ coincides with the fundamental group of a fiber (Seifert) surface of $T_{p,q}$, i.e. the free group of rank $(p-1)(q-1)/2$. $\square$

Question 3.2. *Let K be a nontrivial torus knot. Suppose that g does not belong to the commutator subgroup $[G(K), G(K)]$. Then is the number of slopes r_i such that $g \in \langle\langle r_i \rangle\rangle$ finite?*

4. Elements in knot groups having high tolerance to Dehn fillings

Let K be a nontrivial knot in S^3. Recall that $\langle\langle 1/0 \rangle\rangle$, the normal closure of a meridian, coincides with the knot group $G(K)$ itself.

Proposition 4.1. *Let μ be a meridian of a nontrivial knot K. Then $\mu \in \langle\langle r \rangle\rangle$ if and only if $r = 1/0$.*

Proof. It is obvious that $\mu \in \langle\langle 1/0 \rangle\rangle$. Suppose that $\mu \in \langle\langle r \rangle\rangle$. Then μ vanishes in $\pi_1(K(r))$, and thus $\pi_1(K(r)) = G(K)/\langle\langle r \rangle\rangle = \langle\langle \mu \rangle\rangle/\langle\langle r \rangle\rangle = \{1\}$. Then Property P [13] implies $r = 1/0$. □

As we mentioned above $\langle\langle 1/0 \rangle\rangle = G(K)$, but for each $r \in \mathbb{Q}$, the normal closure $\langle\langle r \rangle\rangle$ is not so large in the sense that $\bigcup_{r\in\mathbb{Q}} \langle\langle r \rangle\rangle$ does not cover $G(K)$. By considering conjugates of the meridian μ, we can easily prove:

Theorem 4.2. *$G(K) - \bigcup_{r\in\mathbb{Q}} \langle\langle r \rangle\rangle$ contains infinitely many elements.*

Proof. By Proposition 4.1 the meridian μ belongs to $G(K) - \bigcup_{r\in\mathbb{Q}} \langle\langle r \rangle\rangle$. Note that for any element $g \in G(K)$, $g^{-1}\mu g \in \langle\langle r \rangle\rangle$ if and only if $r = 1/0$, and hence $g^{-1}\mu g \in G(K) - \bigcup_{r\in\mathbb{Q}} \langle\langle r \rangle\rangle$ as well. To prove the proposition, it is sufficient to show that $H = \{g^{-1}\mu g \mid g \in G(K)\}$ is an infinite set.

Let $C(\mu)$ be the centralizer of μ in $G(K)$. Then it follows from [2, Theorems 2.5.1 and 2.5.2] ([11, II.4.7.Proposition] and [12, p.82]) that $C(\mu)$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. We now observe that $C(\mu)$ has infinite index in $G(K)$. Suppose for a contradiction that $C(\mu)$ has finite index in $G(K)$. Take a finite cover $\widetilde{E(K)}$ of $E(K)$ associated to the subgroup $C(\mu)$. Since $\widetilde{E(K)}$ is boundary-irreducible, $C(\mu)$ is not $\mathbb{Z}$. Assume for a contradiction that $C(\mu) \cong \mathbb{Z} \oplus \mathbb{Z}$. Then $G(K)$ has the finite index subgroup $C(\mu) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is the fundamental group of the torus. Hence [9, 10.6.Theorem] shows that $E(K)$ is $S^1 \times S^1 \times [0, 1]$ or the twisted I–bundle over the Klein bottle. In either case we have a contradiction. Thus $C(\mu)$ has infinite index in $G(K)$, and $C(\mu)\backslash G(K)$ is an infinite set. Take representatives $g_1, g_2, \dots$ so that g_i and g_j belong to distinct right cosets if $i \neq j$. Then $\{g_i^{-1}\mu g_i\}_{i\in\mathbb{N}}$ is an infinite set. In fact, if $g_i^{-1}\mu g_i = g_j^{-1}\mu g_j$ for some $i \neq j$, then $\mu g_i g_j^{-1} = g_i g_j^{-1}\mu$, which means that $g_i g_j^{-1} \in C(\mu)$. Hence g_i and g_j belong to the same right coset, contradicting the choice of elements g_i's. □

5. Length of closed geodesics and the number of vanishing slopes

Let X be a compact, orientable 3–manifold whose boundary consists of a single torus T and interior int$X$ admits a complete hyperbolic structure of finite volume, and $C \cong S^1 \times S^1 \times [0, \infty)$ a cusp neighborhood in int$X$ with boundary ∂C, naturally identified with T, endowed with the Euclidian metric induced from the hyperbolic metric on int$X$. Theorem 2.1 says that for each homotopically essential loop $c \subset X$, there are only finitely many slopes r on T such that c becomes null-homotopic in $X(r)$. However, Theorem 2.1 does not say the number of such slopes. In this section we will discuss the number of such slopes from a viewpoint of hyperbolic geometry. Let us recall the following result due to Agol [1, Section 8].

Theorem 5.1. *Let γ be a closed geodesic in* $\mathrm{int}X$ *disjoint from C. If γ becomes null-homotopic after Dehn fillings on X along slopes r and r', then the distance Δ between r and r' is at most* $36/\mathrm{Area}(\partial C)$.

PROOF. Suppose that γ becomes null-homotopic after Dehn fillings on X along slopes r and r'. By the proof of Theorem 2.1, if $\gamma \cap C = \emptyset$, then the Euclidean lengths of r and r' on ∂C are both at most 6. Then it is well-known that the distance Δ between r and r' satisfies $\Delta \leq 36/\mathrm{Area}(\partial C)$. See [1, Section 8] for example. $\square$

In general, closed geodesics γ (which represent nontrivial elements in $G(K)$) intersect C.

Proposition 5.2. *Let γ be a closed geodesic in* $\mathrm{int}X$ *with hyperbolic length ℓ less than* 0.107*. If*

$$\mathrm{Area}(\partial C) < \frac{1}{2}\sqrt{t - \frac{1}{t}}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) \tag{5.1}$$

with

$$t := e^R,\ R := \sinh^{-1}\left(\sqrt{\frac{1}{2}\left(\frac{\sqrt{1-2k}}{k} - 1\right)}\right),\ k := \cosh\left(\sqrt{\frac{4\pi\ell}{\sqrt{3}}}\right) - 1,$$

then γ does not intersect C.

PROOF. By [15, Theorem in page 1046], for a geodesic γ in a complete hyperbolic 3–manifold of length ℓ less than 0.107, there exists an embedded solid tube around γ whose radius R given in the proposition. Together with the next lemma, the proposition can be shown from this fact by considering the lifts of γ and ∂C to the universal covering space $\mathbb{H}^3$. $\square$

Lemma 5.3. *Let γ be a geodesic in hyperbolic* 3*–space $\mathbb{H}^3$, and H the horosphere tangent to γ. Let B_R denote the tube neighborhood of γ with radios R, i.e., $\{p \in \mathbb{H}^3 \mid d(p, \gamma) \leq R\}$. Then, with $t := e^R$, we have*

$$\mathrm{Area}(B_R \cap H) \geq \frac{1}{2}\sqrt{t - \frac{1}{t}}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right).$$

PROOF. As in Figure 5.1, we work with the upper half space model of $\mathbb{H}^3$, and identify $\partial\mathbb{H}^3$ as the complex plane $\mathbb{C} = \{x + yi \mid x, y \in \mathbb{R}\}$, where i denotes the imaginary unit. Without loss of generality, we may assume that γ has the endpoints 1 and -1 on $\mathbb{C} = \partial\mathbb{H}^3$, and H is the horizontal plane of height 1, i.e., the plane defined by $z = 1$.

The idea to obtain a lower bound of the area of the region $B_R \cap H$ is to consider the rhombus inscribing to $B_R \cap H$. As usual, we regard H as the Euclidean plane. Let P (resp. P') be the intersection point of the xz-plane

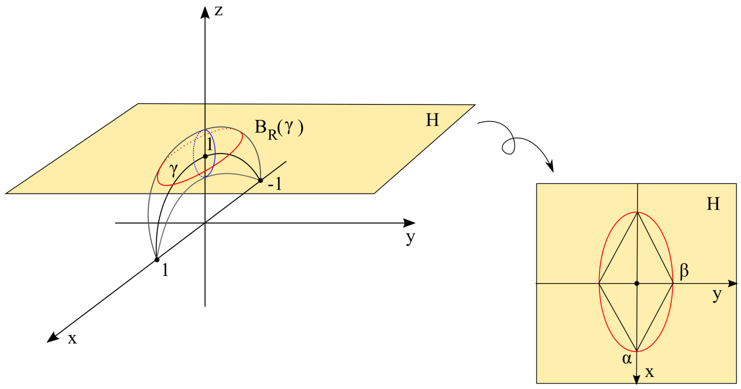

Figure 5.1:

(resp. yz-plane) and $\partial(B_R \cap H)$, and α (resp. β) be the Euclidean distance from the origin of H, i.e. $(0,0,1)$ in $\mathbb{H}^3$, to P (resp. P').

To compute the value α, we regard the xz-plane as the complex plane $\mathbb{C}$ with setting the x-axis as the real axis. On that plane, we use the Möbius transformation φ keeping 1, -1 and mapping a point on the imaginary axis to the point P. Note that the preimage under φ of the geodesic containing P and perpendicular to γ is the imaginary axis with $z > 0$. Then the point mapped to P corresponds to ti with $t = e^R$, for the hyperbolic distance between P and γ is R. See Figure 5.2(i).

Recall that the matrix representing the Möbius transformation keeping 1 and -1 is given by the matrix such as $\begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}$ for some $x \in \mathbb{R}$. Since φ maps ti to the point P, we have $\mathrm{Im}\,(\varphi(ti)) = \mathrm{Im}(P) = 1$, equivalent to

$$\mathrm{Im}\left(\frac{(\cosh(x_0))ti + \sinh(x_0)}{(\sinh(x_0))ti + \cosh(x_0)}\right) = 1$$

for some particular $x_0 \in \mathbb{R}$.

This is equivalent to $t^2 \sinh^2(x_0) + \cosh^2(x_0) = t$, and by solving this, we see that

$$\sinh^2(x_0) = \frac{t-1}{t^2+1}, \ \cosh^2(x_0) = \frac{t^2+t}{t^2+1} \tag{5.2}$$

Now, since $\alpha = \mathrm{Re}\,(\varphi(ti))$, we obtain that

$$\alpha = \mathrm{Re}\left(\frac{(\cosh(x_0))ti + \sinh(x_0)}{(\sinh(x_0))ti + \cosh(x_0)}\right) = \frac{1+t^2}{t}\cosh(x_0)\sinh(x_0).$$

This implies that $\alpha = \sqrt{t - \frac{1}{t}}$.

On the other hand, to compute β, we consider the yz-plane, which we regard it as $\mathbb{C}$ with setting the y-axis as the real axis. On that plane, we use the Möbius transformation ψ keeping i and mapping a point on the imaginary axis to the point P'. Note that the preimage under ψ of the geodesic containing i and P' is the imaginary axis with $z > 0$. Then the point mapped to P' corresponding to ti with $t = e^R$, for the hyperbolic distance between P' and γ is R. See Figure 5.2(ii).

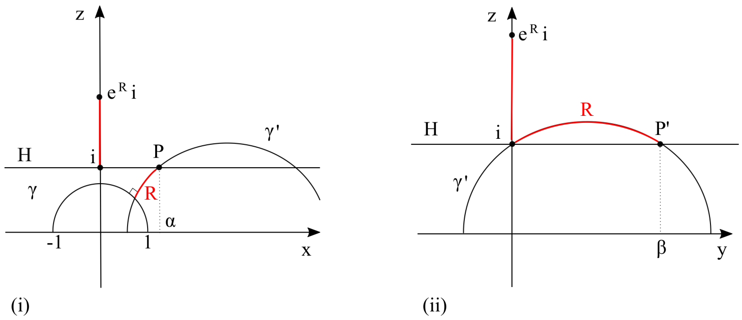

Figure 5.2:

Recall that the matrix representing the Möbius transformation keeping i invariant is described by the matrix such as $\begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}$ for some $x \in \mathbb{R}$. Since ψ maps ti to the point P', we have $\mathrm{Im}\,(\psi(ti)) = \mathrm{Im}(P') = 1$, that is,

$$\mathrm{Im}\left(\frac{(\cos(x_0'))ti - \sin(x_0')}{(\sin(x_0'))ti + \cos(x_0')}\right) = 1$$

for some particular $x_0' \in \mathbb{R}$.

In the same way as above, by solving $t^2 \sin^2(x_0') + \cos^2(x_0') = t$, we see that

$$\sin^2(x_0') = \frac{1}{t+1}, \quad \cos^2(x_0') = \frac{t}{t+1} \tag{5.3}$$

Now, since $\beta = \mathrm{Re}\,(\psi(ti))$, we obtain that

$$\beta = \mathrm{Re}\left(\frac{(\cos(x_0'))ti - \sin(x_0')}{(\sin(x_0'))ti + \cos(x_0')}\right) = \frac{t^2-1}{t}\cos(x_0')\sin(x_0').$$

This implies that $\beta = \sqrt{t} - \frac{1}{\sqrt{t}}$.

Consequently we obtain that

$$\mathrm{Area}(B_R \cap H) \geq \frac{\alpha\beta}{2} = \frac{1}{2}\sqrt{t - \frac{1}{t}}\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)$$

as desired. □

Corollary 5.4. *Let K be any hyperbolic knot in S^3. Let γ be a closed geodesic in $S^3 - \mathcal{N}(K)$with length ≤ 0.0088. Then γ cannot be null-homotopic after all but 12 Dehn fillings.*

Proof. By direct calculations using Proposition 5.2, we see that if $\ell \leq 0.0088$, then any closed geodesic γ of length ℓ in int$X$ cannot intersect a cusp neighborhood with boundary of area at most 3.353. On the other hand, it is known by [5, Proposition 5.8] that every orientable 1-cusped hyperbolic 3–manifold contains a cusp neighborhood with boundary of area at least 3.35. Therefore a closed geodesic of length at most 0.0088 become null-homotopic after a pair of Dehn fillings only if they are along slopes of distance at most $36/3.35 < 10.75$ by Theorem 5.1. Then, by [1, Lemma 8.2], the number of the slopes each pair of which has distance at most 10 is at most 12, and so, any closed geodesic of length at most 0.0088 cannot become null-homotopic after all but 12 Dehn fillings. □

Example 5.5. *Let K be a knot in S^3 and c a trivial knot such that $K \cup c$ is a hyperbolic link in S^3. Perform n–twisting along c so that $|n|$ is sufficiently large to obtain a knot $K_n \subset S^3$. Then the image of c under this twisting becomes a closed geodesic in $S^3 - \mathcal{N}(K)$ with length ≤ 0.0088 [18, 19]. So c cannot be null-homotopic after all but 12 Dehn fillings on K_n.*

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