Seifert fibered surgeries with distinct primitive/Seifert positions

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Abstract

We call a pair (K, m) of a knot K in the 3–sphere S^3 and an integer m a Seifert fibered surgery if m–surgery on K yields a Seifert fiber space. For most known Seifert fibered surgeries (K, m), K can be embedded in a genus 2 Heegaard surface of S^3 in a primitive/Seifert position, the concept introduced by Dean as a natural extension of primitive/primitive position defined by Berge. Recently Guntel has given an infinite family of Seifert fibered surgeries each of which has distinct primitive/Seifert positions. In this paper we give yet other infinite families of Seifert fibered surgeries with distinct primitive/Seifert positions from a different point of view.

Key words: Dehn surgery, Seifert fiber space, primitive/Seifert position
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1 Introduction

Let (K, m) be a pair of a knot K in S^3 and an integer m, and denote by $K(m)$ the manifold obtained from S^3 by m–surgery on K. We say that (K, m) is a *Seifert fibered surgery* if $K(m)$ is a Seifert fiber space. We regard that two Seifert fibered surgeries (K, m) and (K', m') are the *same* if K has the same knot type as K' (i.e. K is isotopic to K' in S^3) and $m = m'$. For a genus 2 handlebody H and a simple closed curve c in ∂H, we denote H with a 2–handle attached along c by $H[c]$.

Let $S^3 = V \cup_F W$ be a genus 2 Heegaard splitting of S^3, i.e. V and W are genus 2 handlebodies in S^3 with $V \cap W$ a genus 2 Heegaard surface F. It is known that such a splitting is unique up to isotopy in S^3 [17]. We say that a Seifert fibered surgery (K, m) has a *primitive/Seifert position* (F, K', m) if K' is a simple closed curve in a genus 2 Heegaard surface F such that $K'(\subset S^3)$ has the same knot type as K and satisfies the following three conditions.

- K' is *primitive* with respect to V, i.e. $V[K']$ is a solid torus.
- K' is *Seifert* with respect to W, i.e. $W[K']$ is a Seifert fiber space with the base orbifold $D^2(p, q)$ $(p, q \geq 2)$.
- The *surface slope* of K' with respect to F (i.e. the isotopy class in $\partial N(K')$ represented by a component of $\partial N(K') \cap F$) coincides with the surgery slope m.

For the primitive/Seifert position (F, K', m) above, we define the *index set* $i(F, K', m)$ to be the set $\{p, q\}$.

In general, if a knot K in S^3 has a primitive/Seifert position with surface slope m, then K is strongly invertible ([14, Claim 5.3]) and $K(m) \cong V[K] \cup W[K]$ is a Seifert fiber space or a connected sum of lens spaces. In particular, if K is hyperbolic, then the latter case cannot happen by the positive solution to the cabling conjecture for strongly invertible knots [4].

The notion of primitive/Seifert position was introduced by Dean [2] as a natural modification of Berge's primitive/primitive position [1]. It is conjectured that all the lens surgeries have primitive/primitive positions [7]. On the other hand, there are infinitely many Seifert fibered surgeries with no primitive/Seifert positions [11,3,16]; nevertheless the majority of Seifert fibered surgeries have such positions. Let (K, m) be a Seifert fibered surgery with two primitive/Seifert positions (F_1, K_1, m) and (F_2, K_2, m). Then, we say that (F_1, K_1, m) and (F_2, K_2, m) are the *same* if there is an orientation preserving homeomorphism f of S^3 such that $f(F_1) = F_2$ and $f(K_1) = K_2$; otherwise, they are *distinct*. It is natural to ask whether a Seifert fibered surgery (K, m) can have distinct primitive/Seifert positions. Recently Guntel [8] has given an infinite family of such examples. Her examples are twisted torus knots studied by Dean [2]. Among them, she finds infinitely many pairs of knots K_1, K_2 which have primitive/Seifert positions with the same surface slopes, and shows that K_1, K_2 are actually the same as knots in S^3, but their primitive/Seifert positions are distinct.

Theorem 1.1 ([8]) *There exist infinitely many Seifert fibered surgeries each of which has distinct primitive/Seifert positions.*

Remark 1.2 In Theorem 1.1, we can choose a Seifert fibered surgery (K, m) with distinct primitive/Seifert positions so that K is a hyperbolic knot whose complement $S^3 - K$ has an arbitrarily large volume.

In the present paper, we give yet other families of Seifert fibered surgeries with distinct primitive/Seifert positions from a different point of view. Our examples are twisted torus knots studied in [13,14] (Theorem 2.1), and also Seifert fibered surgeries constructed by the Montesinos trick in [5,6] (Theorem 3.3). We find infinitely many knots such that each knot K lies in two genus 2 Heegaard surfaces F_1, F_2 with the same surface slopes m, and (F_1, K, m) and (F_2, K, m) are distinct primitive/Seifert positions.

We use Lemma 1.3 to show that two primitive/Seifert positions are distinct.

Lemma 1.3 *Two primitive/Seifert positions (F_1, K_1, m) and (F_2, K_2, m) for a Seifert fibered surgery (K, m) are distinct if $i(F_1, K_1, m) \neq i(F_2, K_2, m)$.*

Proof of Lemma 1.3. Let us denote the Heegaard splitting of S^3 given by F_1 (resp. F_2) by $V \cup_{F_1} W$ (resp. $V' \cup_{F_2} W'$). We may assume that $V[K_1]$ (resp. $V'[K_2]$) is a solid torus, and $W[K_1]$ (resp. $W'[K_2]$) is a Seifert fiber space with the base orbifold $D^2(p, q)$ (resp. $D^2(p', q')$). Suppose for a contradiction that we have an orientation preserving homeomorphism f of S^3 such that $f(K_1) = K_2$ and $f(F_1) = F_2$. Then there are two cases to consider: $f(V) = V'$ or $f(V) = W'$. In the former case $f(W) = W'$ and we have also an orientation preserving homeomorphism $f_W : W[K_1] \to W'[K_2]$. This then implies that $\{p, q\} = \{p', q'\}$, i.e. $i(F_1, K_1, m) = i(F_2, K_2, m)$. This is a contradiction. In the latter $f(W) = V'$ and we have an orientation preserving homeomorphism $f_V : V[K_1] \to W'[K_2]$. However, this is impossible because $V[K_1]$ is a solid torus and $W'[K_2]$ is a Seifert fiber space over the base orbifold $D^2(p', q')$ $(p', q' \geq 2)$. $\square$(Lemma 1.3)

2 Seifert fibered surgeries which have distinct primitive/Seifert positions I

Let V_1 be a standardly embedded solid torus in S^3; denote the solid torus $S^3 - \mathrm{int}V_1$ by V_2. Let $T_{p,q}$ be a torus knot which lies in ∂V_1 and wraps p times meridionally and q times longitudinally in V_1. Take a trivial knot $c_{p,q}$ in $S^3 - T_{p,q}$ as in Figure 1; $c_{p,q} \cap V_i$ consists of a single properly embedded arc in V_i which is parallel to ∂V_i. Note that the linking number $\mathrm{lk}(T_{p,q}, c_{p,q})$ with orientations indicated in Figure 1 is $p+q$, and that $c_{p,q}$ is a meridian of $T_{p,q}$ if $|p+q| = 1$. So in the following we assume $|p+q| > 1$. We denote by $K(p, q, p+q, n)$ the twisted torus knot obtained from $T_{p,q}$ by twisting n times along $c_{p,q}$. As shown in [12, Claim 9.2] ([3, Theorem 3.19(3)]), $T_{p,q} \cup c_{p,q}$ is a hyperbolic link in S^3. Hence by [3, Proposition 5.11] $K(p, q, p+q, n)$ is a hyperbolic knot if $|n| > 3$. In the following, for simplicity, we denote $c_{p,q}$ by c.

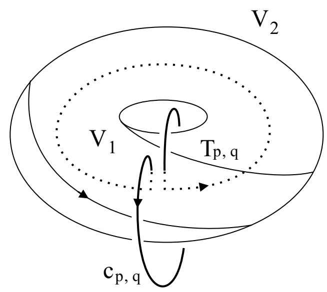


Fig. 1.

In [14] it is shown that $(pq + n(p+q)^2)$–surgery on $K(p, q, p+q, n)$ yields a Seifert fiber space over S^2 with at most three exceptional fibers of indices $|p|, |q|, |n|$. If $n = 0$, then it is a connected sum of two lens spaces, if $n = \pm 1$, then it is a lens space. In fact, as shown in [3], $(K(p, q, p+q, \varepsilon), pq + \varepsilon(p+q)^2)$ is a Berge's lens surgery [1] of Type VII or VIII according as $\varepsilon = 1$ or -1.

Theorem 2.1 *Each Seifert fibered surgery* $(K(p, q, p+q, n),\ pq + n(p+q)^2)$ $(n \neq 0, \pm 1)$ *has distinct primitive/Seifert positions.*

The proof of Corollary 4.8 in [3] shows that for any r there are p and q such that for infinitely many n, $K(p, q, p+q, n)$ is a hyperbolic knot whose complement in S^3 has volume greater than r. Hence, Theorem 2.1 implies Theorem 1.1 and Remark 1.2.

Proof of Theorem 2.1. We follow the argument given in the proof of [14, Proposition 5.2]. Let us put $\tau_i = c \cap V_i$ $(i = 1, 2)$; $c = \tau_1 \cup \tau_2$. Then $H_1 = V_1 - \mathrm{int}N(c)$ and $H_2 = V_2 - \mathrm{int}N(c)$ are genus 2 handlebodies, and $T_{p,q}$ lies on $\partial V_i - \mathrm{int}N(c) = H_1 \cap H_2$. Note that $H_1 \cup H_2 = S^3 - \mathrm{int}N(c)$; see Figure 2.

We denote by U the solid torus glued to $S^3 - \mathrm{int}N(c)$ to construct the surgered

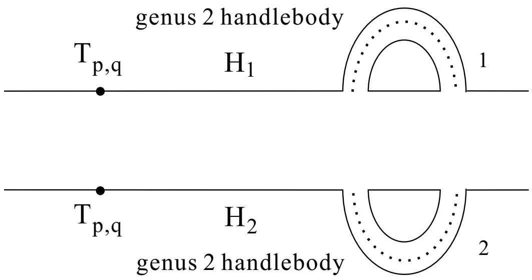


Fig. 2.

manifold $c(-\frac{1}{n})$. Let $A_i = U \cap H_i$; $A_i(\subset \partial H_i)$ is an annulus whose core is a meridian of c. Since $\tau_i(\subset V_i)$ is parallel to ∂V_i, there is a disk Δ_i in H_i such that $\partial\Delta_i$ is the union of an arc in the annulus A_i and an arc in $\partial H_i - \mathrm{int}A_i$. Note that $N(\Delta_i) \cup U$ and the closure of $H_i - N(\Delta_i)$ are solid tori, and their intersection is a disk. This implies that $H_i \cup U = (N(\Delta_i) \cup U) \cup (H_i - N(\Delta_i))$ is a genus 2 handlebody for $i = 1, 2$.

Lemma 2.2 $(H_1 \cup U) \cup_F H_2$ *and* $H_1 \cup_{F'} (H_2 \cup U)$ *are both genus* 2 *Heegaard splitting of* $S^3 = c(-\frac{1}{n})$, *where* $F = \partial(H_1 \cup U) = \partial H_2$ *and* $F' = \partial(H_2 \cup U) = \partial H_1$.

Let $\{\mu, \lambda\}$ be a meridian-longitude basis for $H_1(\partial N(c))$. Then, a meridian and thus a longitude of U represent $-n\lambda + \mu$ and λ in $H_1(\partial N(c))$, respectively. It follows that a meridian of $N(c)$ winds U n times longitudinally. We thus have the following.

Lemma 2.3 *The core of the annulus* $A_i(\subset \partial U)$ *winds* U n *times longitudinally.*

The twisted torus knot $K(p, q, p+q, n)$ lies on F and F'. See Figure 3. In either case, the surface slope of $T_{p,q} = K(p, q, p+q, 0)$ is pq and the surface slope of $K(p, q, p+q, n)$ is the image of that of $T_{p,q}$ under n-twisting along c. Since $\mathrm{lk}(T_{p,q}, c) = p + q$, the surface slope of $K(p, q, p+q, n)$ is $pq + n(p+q)^2$.

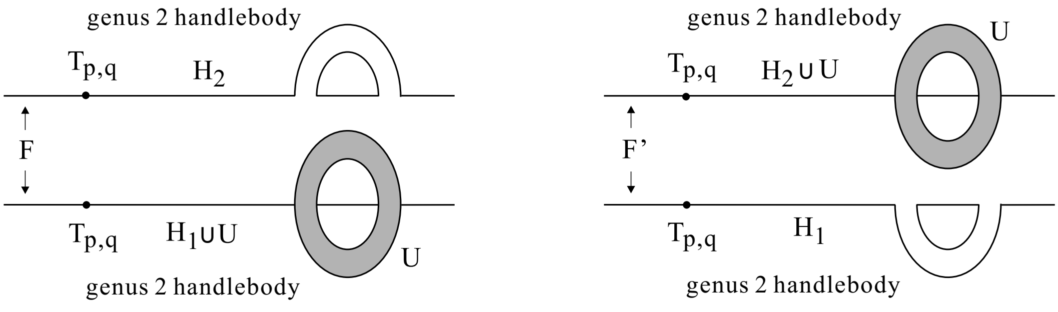


Fig. 3. Genus 2 Heegaard splittings carrying $T_{p,q}$

Lemma 2.4 (1) $H_1[T_{p,q}]$ *is a fibered solid torus in which the core is an exceptional fiber of index* $|q|$ *and the core of* A_1 *is a regular fiber.*

(2) $H_2[T_{p,q}]$ *is a fibered solid torus in which the core is an exceptional fiber of index* $|p|$ *and the core of* A_2 *is a regular fiber.*

Proof of Lemma 2.4. See Lemma 9.1 in [12]. $\square$(Lemma 2.4)

Lemma 2.5 (1) $(H_1 \cup U)[K(p,q,p+q,n)]$ *is a Seifert fiber space over* D^2 *with two exceptional fibers of indices* $|q|$, $|n|$.

(2) $(H_2 \cup U)[K(p,q,p+q,n)]$ *is a Seifert fiber space over* D^2 *with two exceptional fibers of indices* $|p|$, $|n|$.

Proof of Lemma 2.5. First observe that $(H_1 \cup U)[K(p,q,p+q,n)] = H_1[T_{p,q}] \cup U$. Since a regular fiber of $H_1[T_{p,q}]$ contained in A_1 winds U n times longitudinally by Lemmas 2.3 and 2.4(1), $H_1[T_{p,q}] \cup U$ is a Seifert fiber space over D^2 with two exceptional fibers of indices $|q|, |n|$ as claimed in assertion (1). Assertion (2) follows in a similar fashion. $\square$(Lemma 2.5)

Therefore the Seifert fibered surgery $(K(p,q,p+q,n), pq + n(p+q)^2)$ has primitive/Seifert positions in two ways.

(1) $K(p,q,p+q,n)$ is primitive with respect to H_2 and Seifert with respect to $H_1 \cup U$; see Figure 4(i).

(2) $K(p,q,p+q,n)$ is primitive with respect to H_1 and Seifert with respect to $H_2 \cup U$; see Figure 4(ii).

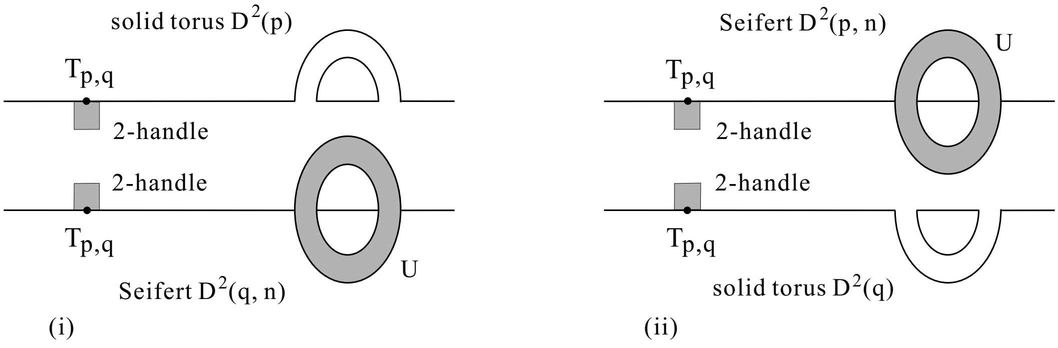


Fig. 4.

To complete the proof of Theorem 2.1 let us show that $(F, K(p,q,p+q,n), pq+n(p+q)^2)$ and $(F', K(p,q,p+q,n), pq+n(p+q)^2)$ are distinct primitive/Seifert positions. The former has index $\{|q|, |n|\}$ by Lemma 2.5(1), and the latter has index $\{|p|, |n|\}$ by Lemma 2.5(2). Since p and q are relatively prime, $|p| \neq |q|$. Then, by Lemma 1.3 they are distinct. $\square$(Theorem 2.1)

Remark 2.6 Twisted torus knots $K(p,q,r,n)$ are obtained from torus knots $T_{p,q}$, roughly speaking, by twisting r strands of $T_{p,q}$ n times. In [2, Theorem 4.1], Dean obtains five classes of $K(p,q,r,\pm 1)$, where $p \geq 0, q \geq 0, 0 \leq r \leq p+q$, with primitive/Seifert positions. Let K be a knot in the classes. We can define H_1, H_2, U for K as for $K(p,q,p+q,n)$ above; then K is contained in two genus 2 Heegaard surfaces $F = \partial(H_1 \cup U), F' = \partial H_1$. Although $K(p,q,p+q,n)$ is primitive/Seifert with respect to both F and F', in general K is primitive/Seifert with respect to F only.

3 Seifert fibered surgeries which have distinct primitive/Seifert positions II

A *tangle* (B,t) is a pair of a 3–ball B and two disjoint arcs t properly embedded in B. A tangle (B,t) is a *rational tangle* if there is a pairwise homeomorphism from (B,t) to the trivial tangle $(D^2 \times [0,1], \{x_1, x_2\} \times [0,1])$ where D^2 is the unit disk and x_1 and x_2 are distinct points in $\mathrm{int} D^2$. Two rational tangles (B,t) and (B,t') are *equivalent* if there is a pairwise homeomorphism $h : (B,t) \to (B,t')$ such that $h|_{\partial B} = \mathrm{id}$. We can construct rational tangles from sequences of integers $[a_1, a_2, \ldots, a_n]$ as shown in Figure 5. Denote by $R(a_1, a_2, \ldots, a_n)$ the associated rational tangle. Each rational tangle can be parametrized by $r \in \mathbb{Q} \cup \{\infty\}$, where the rational number r is given by the continued fraction below. Thus we denote the rational tangle corresponding to r by $R(r)$.

$$r = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots \, + \cfrac{1}{a_1}}}$$

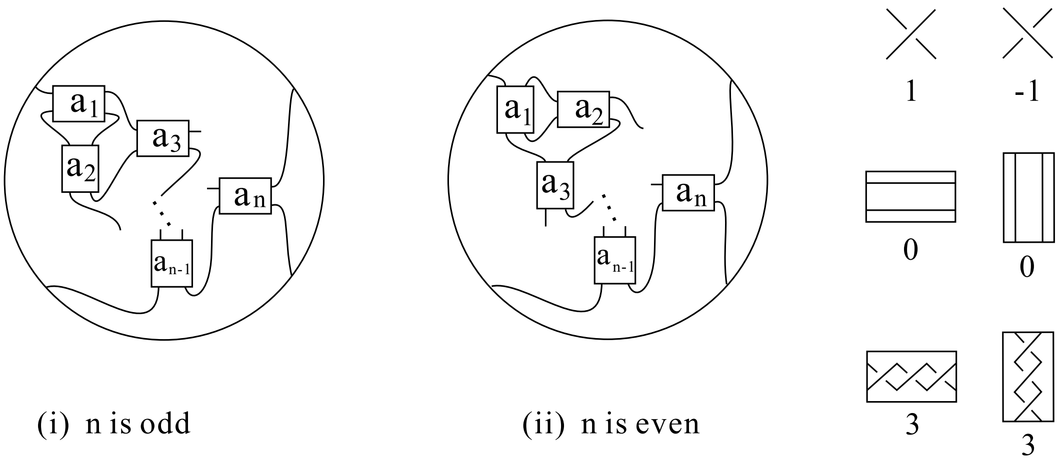
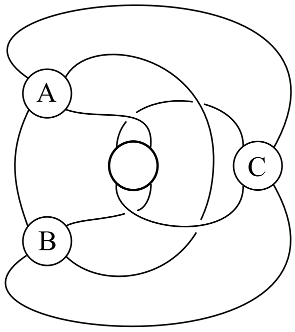
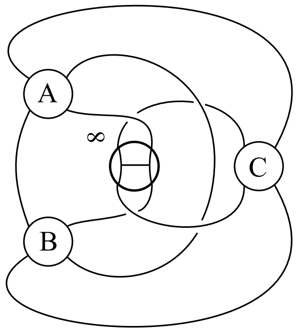


Fig. 5. Rational tangles

Let us consider the tangle $\mathcal{B}(A,B,C)$ given by Figure 6. In [6], the same tangle $\mathcal{B}(A,B,C)$ is defined by Figure 9(a) in [6]. However, the figure contains errors; four crossings of Figure 9(a) in [6] should be reversed. Figure 6 is the corrected diagram. The union of the tangle $\mathcal{B}(A,B,C) = (B_1, t_1)$ and a rational tangle $R(s) = (B_2, t_2)$ gives a pair $(S^3, \tau_s) = (B_1 \cup B_2, t_1 \cup t_2)$. We obtain τ_s, a knot or a link in S^3. In Figure 6 we illustrate the union of $\mathcal{B}(A,B,C)$ and $R(\infty)$.



$\mathcal{B}$(A, B, C)



Union of $\mathcal{B}$(A, B, C) and R(∞)

Fig. 6.

In the following, we assume that τ_∞ is a trivial knot in S^3. Let $\pi_s : \widetilde{S^3}(s) \to S^3$ be the two-fold branched covering of S^3 along τ_s. Since τ_∞ is trivial, $\widetilde{S^3}(\infty) = S^3$. For a subset X of S^3, we often denote $\pi_s^{-1}(X)$ by $\widetilde{X}(s)$, and $\widetilde{X}(\infty)$ by $\widetilde{X}$ for simplicity. Let κ be an arc connecting the two vertical strings of $R(\infty)$ as the horizontal arc in Figure 6. Then the preimage $\pi_\infty^{-1}(\kappa)$ is a knot in S^3; we denote $\pi_\infty^{-1}(\kappa)$ by $k(A,B,C)$. Since the two-fold branched covering of B_2 along the rational tangle t_2 is a solid torus, $\widetilde{B_2}(s)$ is a solid torus and in particular $\widetilde{B_2}$ is a tubular neighborhood of $k(A,B,C)$ in $\widetilde{S^3} = S^3$. Hence $\widetilde{S^3}(s)$ is obtained from S^3 by a Dehn surgery on $k(A,B,C)$. We denote the surgery slope by γ_s. For $(B_2, t_2) = R(s)$, if a properly embedded disk D in $B_2 - t_2$ separates the components of t_2, then $\widetilde{D}(s) = \pi_s^{-1}(D)$ consists of two meridian disks of the glued solid torus $\widetilde{B_2}(s)$. Hence, a component of $\partial\widetilde{D}(s) = \partial\widetilde{D}$ in $\partial\widetilde{B_2}$ represents the surgery slope γ_s.

Although Figure 9(a) in [6] contains errors as mentioned above, Lemma 5.1 in [6] is correct and we have:

Lemma 3.1 (Lemma 5.1 in [6]) *τ_∞ is a trivial knot in S^3 if either* (1) *or* (2) *below holds, where l, m, n, p are integers. The solutions are the only ones, up to interchanging A and B; note that there is a rotation interchanging them.*

(1) $A = R(l), B = R(m, -l), C = R(-n, 2, m-1, 2, 0)$
(2) $A = R(l), B = R(p, -2, m, -l), C = R(m-1, 2, 0)$

In case (1), we denote $k(A,B,C)$ by $k(l,m,n,0)$. In case (2), we denote $k(A,B,C)$ by $k(l,m,0,p)$. As shown in [5,6], $k(A,B,C)$ are mostly hyperbolic knots. See [5,6] for details.

The links τ_0 and τ_1 are Montesinos links with three branches indicated by the 3–balls B_A, B_B, B_C in Figures 7 and 8. Hence, $\widetilde{S^3}(s) = k(A,B,C)(\gamma_s)$, where $s = 0, 1$, is a Seifert fiber space whose exceptional fibers are the cores of $\widetilde{B_A}(s), \widetilde{B_B}(s), \widetilde{B_C}(s)$. Compute the rational numbers corresponding to the rational tangles $(B_A, B_A \cap \tau_s), (B_B, B_B \cap \tau_s), (B_C, B_C \cap \tau_s)$ such that A, B, and C satisfy (1) or (2) in Lemma 3.1; then, we obtain the indices of exceptional fibers of $k(A,B,C)(\gamma_s)$ as follows. If $(B_X, B_X \cap \tau_s)$ where $X \in \{A, B, C\}$ corresponds to a rational number $\frac{p}{q}$, then the Seifert invariant of the core of $\widetilde{B_X}(s)$ is $-\frac{q}{p}$, and the index is $|p|$.

Lemma 3.2 (corrected Proposition 5.4(2), (3), (5), (6) in [6])

(1) (i) *γ_0–surgery on $k(l,m,n,0)$ produces a Seifert fiber space over S^2 with three exceptional fibers, the cores of $\widetilde{B_A}(0), \widetilde{B_B}(0), \widetilde{B_C}(0)$, of indices $|l-1|, |lm+m-1|, |2mn-m-n+1|$.*

(ii) *γ_1–surgery on $k(l,m,n,0)$ produces a Seifert fiber space over S^2 with three exceptional fibers, the cores of $\widetilde{B_A}(1), \widetilde{B_B}(1), \widetilde{B_C}(1)$, of indices $|l+1|, |lm-m-1|, |2mn-m+n|$.*

(2) (i) *γ_0–surgery on $k(l,m,0,p)$ produces a Seifert fiber space over S^2 with three exceptional fibers, the cores of $\widetilde{B_A}(0), \widetilde{B_B}(0), \widetilde{B_C}(0)$, of indices $|l-1|, |2lmp-lm-lp+2mp-m-3p+1|, |m-1|$.*

(ii) *γ_1–surgery on $k(l,m,0,p)$ produces a Seifert fiber space over S^2 with three exceptional fibers, the cores of $\widetilde{B_A}(1), \widetilde{B_B}(1), \widetilde{B_C}(1)$, of indices $|l+1|, |2lmp-lm-lp-2mp+m-p+1|, |m|$.*

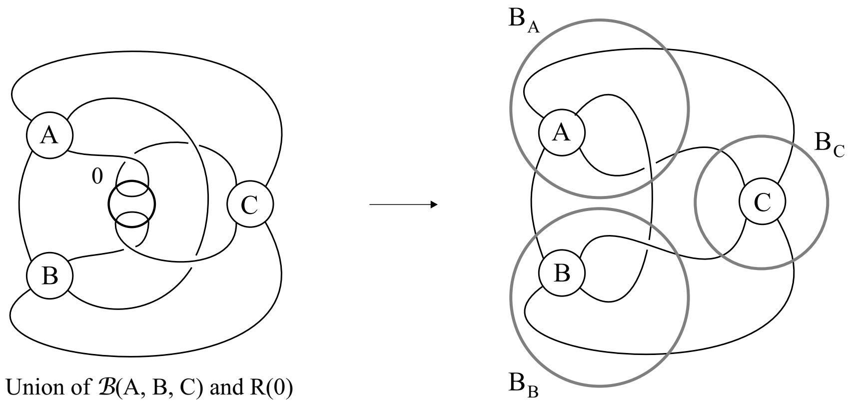


Fig. 7. $\mathcal{B}(A,B,C) \cup R(0)$

In [6] a method is given to find primitive/Seifert positions for Seifert fibered surgeries constructed via tangles and double branched covers, and this is used to show that each of the surgeries $(k(A,B,C), \gamma_s)$ $(s = 0, 1)$ has a primitive/Seifert position. Using this method, we prove that each Seifert fibered surgery $(k(A,B,C),\ \gamma_s)$ $(s = 0, 1)$ has distinct primitive/Seifert positions if the indices of the exceptional fibers which are the cores of $\widetilde{B_A}(s)$ and $\widetilde{B_B}(s)$ are not equal.

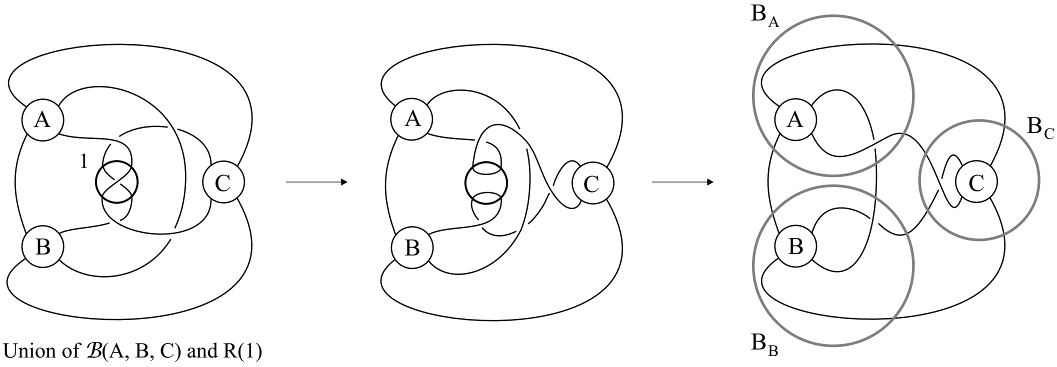


Fig. 8. $\mathcal{B}(A, B, C) \cup R(1)$

Theorem 3.3 *According as* A, B, C *satisfy* (1) *or* (2) *of Lemma* 3.1 *we assume the following.*
If A, B, C *satisfy Lemma* 3.1(1), *assume that* $|l-1| \neq |lm+m-1|$ *if* $s=0$, *and that* $|l+1| \neq |lm-m-1|$ *if* $s=1$.
If A, B, C *satisfy Lemma* 3.1(2), *assume that* $|l-1| \neq |2lmp-lm-lp+2mp-m-3p+1|$ *if* $s=0$, *and that* $|l+1| \neq |2lmp-lm-lp-2mp+m-p+1|$ *if* $s=1$.
Then, each Seifert fibered surgery $(k(A, B, C), \gamma_s)$ $(s=0,1)$ *has distinct primitive/Seifert positions.*

It follows from [6, Proposition 5.6] that the braid index of $k(l, m, n, 0)$ is $2lm-1$ (resp. $2|lm|+1$) if $l>0, m>0$ (resp. $l>0, m<0$), and that of $k(l, m, 0, p)$ is $2lm-l-1$ (resp. $2|lm|+l+1$) if $l>0, m>0$ (resp. $l>0, m<0$). Hence there are infinitely many knots satisfying the conditions in Theorem 3.3.

The assumption in Theorem 3.3 that the indices of the exceptional fibers in $\widetilde{B_A}(s)$ and $\widetilde{B_B}(s)$ are not equal is not a necessary condition for $(k(A, B, C), \gamma_s)$ to have distinct primitive/Seifert positions. Refer to Section 4.

Proof of Theorem 3.3. Let s be 0 or 1. If $s=0$, let S be the 2–sphere in S^3 shown in Figure 9(i), and if $s=1$, let S be the 2–sphere in S^3 shown in Figure 9(ii). In either case let Q_i $(i=1,2)$ be the 3–balls bounded by S as in Figure 9. Note that Figure 9 also describes the union of the tangles $\mathcal{B}(A, B, C) = (B_1, t_1)$ and $R(\infty) = (B_2, t_2)$, and $t_1 \cup t_2 = \tau_\infty$. However, τ_∞ in Figure 9(ii) is obtained by turning back a portion of τ_∞ in Figure 6. The tangles $(Q_i, Q_i \cap \tau_\infty)$ $(i=1,2)$ are 3–string trivial tangles. Hence, the two-fold branched covering $\widetilde{Q_1} \cup \widetilde{Q_2}$ gives a genus 2 Heegaard splitting of $S^3 = \widetilde{S^3}$, and $\widetilde{S} = \widetilde{Q_1} \cap \widetilde{Q_2}$ is a genus 2 Heegaard surface. Note that $S \cap B_2$ is a disk intersecting t_2 transversely in two points and containing the arc κ. This implies that the annulus $\widetilde{S \cap B_2}$ is a tubular neighborhood of the knot $k(A, B, C) = \widetilde{\kappa}$ in the Heegaard surface $\widetilde{S}$. Hence, a component of $\widetilde{S \cap \partial B_2}$ is a simple closed curve in $\partial \widetilde{B_2} = \partial N(k(A, B, C))$ representing the surface slope of $k(A, B, C)$

in the Heegaard surface $\widetilde{S}$.

Now let us show that the surface slope of $k(A,B,C)$ in $\widetilde{S}$ coincides with the surgery slope γ_s. Recall that γ_s–surgery on $k(A,B,C)$ corresponds to replacing $R(\infty)$ with $R(s)$. The disk $S \cap B_2$ in S is, as shown in Figure 9, a "horizontal" disk properly embedded in B_2. If $s=0$ and so S is as in Figure 9(i), then we may assume that $S \cap B_2$ separates the components of t_2 in $R(0)$ after replaced; see Figure 10. It follows that $\widetilde{S \cap \partial B_2}$ in $\partial \widetilde{B_2}$ represents the surgery slope γ_s, so that the surface slope of $k(A,B,C)$ in $\widetilde{S}$ coincides with γ_0 as desired. So assume that $s=1$ and S is as in Figure 9(ii). We need to see that the disk $S \cap B_2$ separates the components of t_2 in $R(1)$ attached to $\mathcal{B}(A,B,C)$ in Figures 6. The first isotopy in Figure 8 turns back a portion of τ_1. Then, in the second figure of Figure 8, t_2 in $R(1)$ becomes horizontal arcs. Hence we may assume that the horizontal disk $S \cap B_2$ in Figure 9(ii) separates the components of t_2 in $R(1)$; see Figure 10. This implies that a component of $\widetilde{S \cap \partial B_2}$ also represents the surgery slope γ_1 as desired.

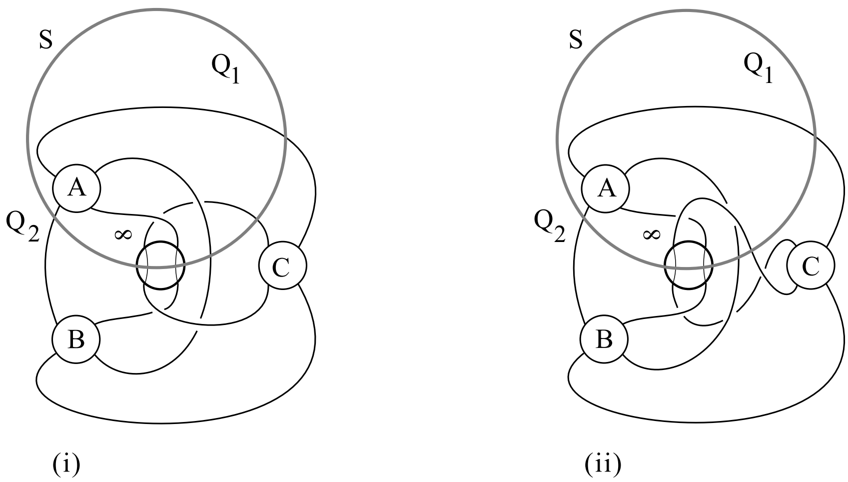
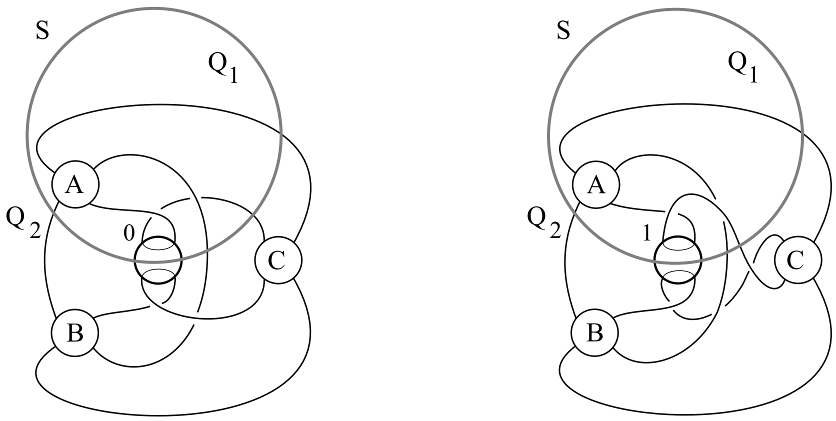


Fig. 9. $\mathcal{B}(A,B,C) \cup R(\infty)$

Lemma 3.4 *The knot $K = k(A,B,C)$ is in a primitive/Seifert position in $\widetilde{S}$ with γ_s $(s=0,1)$ the surface slope, whose index set is the set of indices of exceptional fibers in $\widetilde{S^3}(s)$ corresponding to B_B, B_C.*

Proof of Lemma 3.4. We have already shown that the surface slope of K in $\widetilde{S}$ coincides with the surgery slope γ_s. We show that K is primitive with respect to the genus 2 handlebody $\widetilde{Q_1}$, and Seifert with respect to $\widetilde{Q_2}$. First consider $(S^3, \tau_s) = \mathcal{B}(A,B,C) \cup R(s)$. The 2–sphere S decomposes (S^3, τ_s) into two 2–string tangles $(Q_1, Q_1 \cap \tau_s)$ and $(Q_2, Q_2 \cap \tau_s)$; $(Q_1, Q_1 \cap \tau_s)$ is a rational tangle, and $(Q_2, Q_2 \cap \tau_s)$ is a partial sum of two rational tangles $(B_B, B_B \cap \tau_s)$ and $(B_C, B_C \cap \tau_s)$ in Figures 7, 8. This implies that $\widetilde{Q_2}(s)$ is a Seifert fiber space over the disk whose exceptional fibers are the cores of $\widetilde{B_B}(s), \widetilde{B_C}(s)$.

To complete the proof we prove $\widetilde{Q_i}(s) \cong \widetilde{Q_i}[K]$. We consider $(S^3, \tau_\infty) = \mathcal{B}(A, B, C) \cup R(\infty)$ again. The disk $Q_i \cap \partial B_1$ decomposes Q_i into two 3–balls $Q_i \cap B_1$ and $Q_i \cap B_2$, so that $\widetilde{Q_i}(s) = \widetilde{Q_i \cap B_1}(s) \cup \widetilde{Q_i \cap B_2}(s)$. Note that $B_1 \cap \tau_s = B_1 \cap \tau_\infty$, and τ_s intersects $Q_i \cap B_2$ in an arc whose end points lie in $Q_i \cap \partial B_1$. Hence, $\widetilde{Q_i \cap B_1}(s) = \widetilde{Q_i \cap B_1}$, and $\widetilde{Q_i \cap B_2}(s)$ is a 3–ball attached to $\widetilde{Q_i \cap B_1}$ along the annulus $\widetilde{Q_i \cap \partial B_1}$. In other words, $\widetilde{Q_i}(s)$ is obtained from $\widetilde{Q_i \cap B_1}$ by attaching a 2–handle along the annulus $\widetilde{Q_i \cap \partial B_1}$. Now replacing $R(s)$ with $R(\infty)$ again, let us see the relation between $\widetilde{Q_i}$ and $\widetilde{Q_i \cap B_1}$. It is not difficult to see the pairwise homeomorphism $(Q_i \cap B_2, Q_i \cap \partial B_2, Q_i \cap B_2 \cap \tau_\infty) \cong (D^2 \times [0,1], D^2 \times \{1\}, \{x_1, x_2\} \times [0,1])$, where $x_1, x_2 \in \mathrm{int} D^2$. This shows that $\widetilde{Q_i \cap \partial B_2}$ is a properly embedded annulus in $\widetilde{Q_i}$ parallel to $\widetilde{S \cap B_2}$, a tubular neighborhood of K in $\widetilde{S}$. Hence, there is a pairwise homeomorphism from $(\widetilde{Q_i \cap B_1}, \widetilde{Q_i \cap \partial B_1})$ to $(\widetilde{Q_i}, \widetilde{S \cap B_2})$. This implies $\widetilde{Q_i}(s) \cong \widetilde{Q_i}[K]$ as desired.
$\square$(Lemma 3.4)



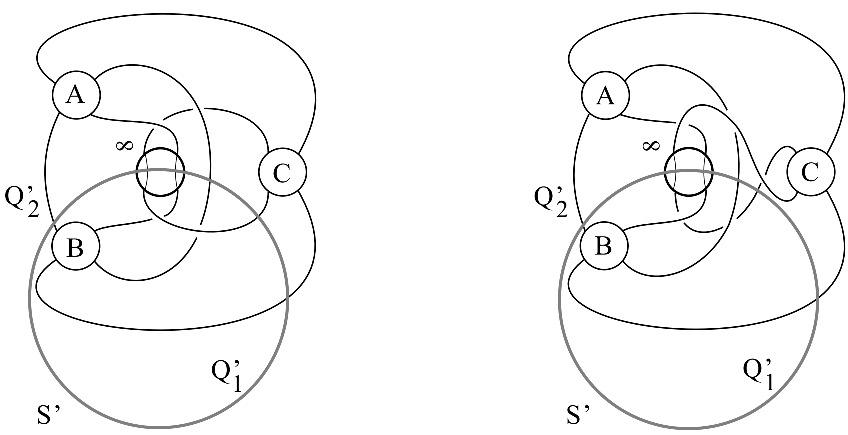
Union of $\mathcal{B}$(A, B, C) and R(0) Union of $\mathcal{B}$(A, B, C) and R(1)

Fig. 10.

To find yet another primitive/Seifert position of $(k(A, B, C), \gamma_s)$, take the 2–sphere S' in S^3 as in (i) or (ii) of Figure 11 according as $s = 0$ or 1. Let Q'_i $(i = 1, 2)$ be the 3–balls bounded by S' as in Figure 11. Then, we can apply the arguments in the first and second paragraphs of the proof of Theorem 3.3 to S', Q'_1, Q'_2 instead of S, Q_1, Q_2. It follows that $\widetilde{Q'_1} \cup \widetilde{Q'_2}$ is a genus 2 Heegaard splitting of $\widetilde{S^3} = S^3$ with $\widetilde{S'}$ the Heegaard surface, and $k(A, B, C)$ is contained in $\widetilde{S'}$ with γ_s $(s = 0, 1)$ the surface slope.

We can also apply most of the arguments in the proof of Lemma 3.4. The only difference is the fact $(Q'_2, Q'_2 \cap \tau_s)$ is a partial sum of $(B_A, B_A \cap \tau_s)$ and $(B_C, B_C \cap \tau_s)$ instead of $(B_B, B_B \cap \tau_s)$ and $(B_C, B_C \cap \tau_s)$; see Figure 12. Therefore, we obtain Lemma 3.5 below.

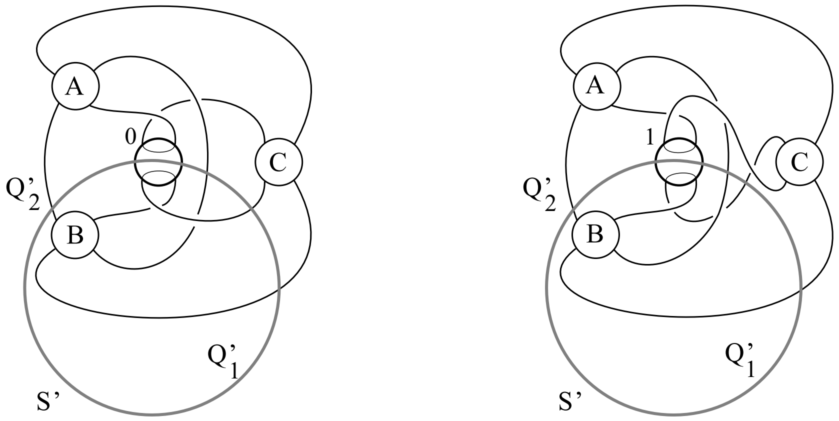
Lemma 3.5 *The knot $k(A, B, C)$ is in a primitive/Seifert position in $\widetilde{S'}$ with*



(i) (ii)

Fig. 11. $\mathcal{B}(A, B, C) \cup R(\infty)$

γ_s $(s = 0, 1)$ *the surface slope, whose index set is the set of indices of exceptional fibers in* $\widetilde{S^3}(s)$ *corresponding to* B_A, B_C.



Union of $\mathcal{B}$(A, B, C) and R(0) Union of $\mathcal{B}$(A, B, C) and R(1)

Fig. 12.

Recall that by the assumption of Theorem 3.3 together with Lemma 3.2, the indices of the exceptional fibers of $\widetilde{S^3}(s) = k(A, B, C)(\gamma_s)$ corresponding to B_A and B_B are not equal. Lemmas 3.4 and 3.5 then imply that the index sets of the primitive/Seifert positions $(\widetilde{S}, k(A, B, C), \gamma_s)$ and $(\widetilde{S'}, k(A, B, C), \gamma_s)$ are not equal. Hence, by Lemma 1.3 these are distinct primitive/Seifert positions for $(k(A, B, C), \gamma_s)$. This completes the proof of Theorem 3.3. $\square$(Theorem 3.3)

Remark 3.6 It follows from Lemma 3.2 that the set $\{k(A, B, C)(\gamma_s)\}$ $(s = 0, 1)$ consists of infinitely many Seifert fiber spaces. If two Seifert fibered surgeries $(k(A, B, C), \gamma_s)$ and $(k(A', B', C'), \gamma'_s)$ are the same, then $k(A, B, C)(\gamma_s)$ and $k(A', B', C')(\gamma'_s)$ are homeomorphic. Thus the set $\{(k(A, B, C), \gamma_s)\}$ contains infinitely many Seifert fibered surgeries.

4 Questions

In Theorems 2.1 and 3.3, Seifert fibered surgeries with distinct primitive/Seifert positions have distinct index sets. However, this is not always the case. Consider the Seifert fibered surgery $(k(2,4,n,0),\gamma_1)$ in Section 3. The result of γ_1–surgery on $K = k(2,4,n,0)$ is a Seifert fiber space with the base orbifold $S^2(\frac{-1}{3},\frac{4}{3},\frac{16n-7}{9n-4})$. Following Lemma 3.4, we see that (K,γ_1) has a primitive/Seifert position $(\widetilde{S},K,\gamma_1)$ such that $\widetilde{Q}_2[K]$ is a Seifert fiber space over the disk with Seifert invariants $\frac{4}{3},\frac{16n-7}{9n-4}$, where $\widetilde{Q}_2$ is a genus 2 handlebody bounded by $\widetilde{S}$. Similarly, Lemma 3.5 shows that (K,γ_1) has a primitive/Seifert position $(\widetilde{S'},K,\gamma_1)$ such that $\widetilde{Q'_2}[K]$ is a Seifert fiber space over the disk with Seifert invariants $\frac{-1}{3},\frac{16n-7}{9n-4}$, where $\widetilde{Q'_2}$ is a genus 2 handlebody bounded by $\widetilde{S'}$. Thus $i(\widetilde{S},K,\gamma_1) = i(\widetilde{S'},K,\gamma_1) = \{3,|9n-4|\}$. If $(\widetilde{S},K,\gamma_1)$ and $(\widetilde{S'},K,\gamma_1)$ were the same, then following the argument in the proof of Lemma 1.3, we would have an orientation preserving homeomorphism from $\widetilde{Q}_2[K]$ to $\widetilde{Q'_2}[K]$; by [10, VI.18.Theorem] the homeomorphism is fiber preserving up to isotopy. However, since $\frac{4}{3}\not\equiv\frac{-1}{3}$ mod 1, there is no such a homeomorphism [9, Proposition 2.1]. Hence the primitive/Seifert positions $(\widetilde{S},k(2,4,n,0),\gamma_1)$ and $(\widetilde{S'},k(2,4,n,0),\gamma_1)$ are distinct.

Question 4.1 Does there exist a Seifert fibered surgery which has distinct primitive/Seifert positions (F_1,K_1,m) and (F_2,K_2,m) satisfying the following condition?
Condition. Let W_i $(i=1,2)$ be a genus 2 handlebody bounded by F_i with respect to which $K_i(\subset\partial W_i)$ is Seifert. Then there is an orientation preserving homeomorphism from $W_1[K_1]$ to $W_2[K_2]$.

Even if a Seifert fibered surgery (K,m) has distinct primitive/Seifert positions, we expect that the number of such positions is not so large. In fact, we do not even have an example of a Seifert fibered surgery which has three primitive/Seifert positions.

Question 4.2 Does there exist a universal bound for the number of primitive/Seifert positions for a Seifert fibered surgery?

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