GENERALIZED TORSION ELEMENTS AND BI-ORDERABILITY OF 3-MANIFOLD GROUPS

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Abstract. It is known that a bi-orderable group has no generalized torsion element, but the converse does not hold in general. We conjecture that the converse holds for the fundamental groups of 3–manifolds, and verify the conjecture for non-hyperbolic, geometric 3–manifolds. We also confirm the conjecture for some infinite families of closed hyperbolic 3–manifolds. In the course of the proof, we prove that each standard generator of the Fibonacci group $F(2, m)$ $(m > 2)$ is a generalized torsion element.

1. Introduction

A group G is said to be *bi-orderable* if G admits a strict total ordering $<$ which is invariant under multiplication from the left and right. That is, if $g < h$, then $agb < ahb$ for any $g, h, a, b \in G$. In this paper, the trivial group $\{1\}$ is considered to be bi-orderable.

Let $g \in G$ be a non-trivial element. If some non-empty finite product of conjugates of g equals to the identity, then g is called a *generalized torsion element*. In particular, any non-trivial torsion element is a generalized torsion element. If a group G is bi-orderable, then G has no generalized torsion element (see Lemma 2.3). In other words, the existence of generalized torsion element is an obstruction for bi-orderability. In the literature [3, 19, 21, 22], a group without generalized torsion element is called an R^*–group or a Γ–torsion-free group. Thus bi-orderable groups are R^*–groups. However, the converse does not hold in general [22, Chapter 4].

If we restrict ourselves to a specific class of groups, say, knot groups or more generally, 3–manifold groups, then we may expect that the converse statement would hold.

Conjecture 1.1. *Let G be the fundamental group of a 3–manifold. Then, G is bi-orderable if and only if G has no generalized torsion element.*

There are several works on the bi-orderability and generalized torsion elements of knot groups. The knot group of any torus knot is not bi-orderable, because it contains generalized torsion elements [23]. Thus Conjecture 1.1 holds for torus knot groups. We remark that the knot exterior of a torus knot is a Seifert fibered manifold. Other examples are twist knots, which have Conway's notation $[2, 2n]$.

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The knot group of a twist knot is bi-orderable if $n > 0$, not bi-orderable if $n < 0$ by [7]. The second named author showed that if $n < 0$, then the knot group contains a generalized torsion element [30]. This means that Conjecture 1.1 holds for twist knot groups as well. Torus knot groups and twist knot groups are one-relator groups, and [6, Question 3] asks whether the conjecture holds for one-relator knot groups, more generally one-relator groups.

We first observe the following, which enables us to restrict our attention to fundamental groups of prime 3–manifolds for Conjecture 1.1.

Proposition 1.2. *Let M be the connected sum of two 3–manifolds M_1 and M_2. Suppose that $G_i = \pi_1(M_i)$ satisfies Conjecture 1.1 for $i = 1, 2$. Then $G = \pi_1(M)$ also satisfies Conjecture 1.1.*

The main purpose of this paper is to confirm Conjecture 1.1 for the fundamental groups of Seifert fibered manifolds, Sol manifolds, which are possibly non-orientable.

Theorem 1.3. *Let M be a compact connected 3–manifold, and let G be its fundamental group. If M is either Seifert fibered or Sol, then G satisfies Conjecture 1.1.*

Any closed geometric 3–manifold which possesses a geometric structure other than a hyperbolic structure is Seifert fibered or admits a Sol structure [28, Theorem 5.1]. Thus Theorem 1.3 shows:

Corollary 1.4. *The fundamental group of any closed, geometric 3–manifold that is non-hyperbolic satisfies Conjecture 1.1.*

The n–fold cyclic branched cover Σ_n of the 3–sphere branched over the figure-eight knot is known to be an L–space and have non-left-orderable fundamental group [9, 26, 29]. In particular, Σ_n is hyperbolic if $n \geq 4$.

Theorem 1.5. *Let Σ_n be the n–fold cyclic branched cover of S^3 over the figure-eight knot. Then $\pi_1(\Sigma_n)$ satisfies Conjecture 1.1.*

Section 3 treats the case where M is a Seifert fibered manifold, and Section 4 examines the case where M is a Sol–manifold. Theorem 1.3 follows from Theorems 3.1 and 4.1. In Section 5 we prove that each generator in the standard cyclic presentation of the Fibonacci group $F(2, m)$ $(m > 2)$ is a generalized torsion element (Theorem 5.2). Since $\pi_1(\Sigma_n)$ is isomorphic to $F(2, 2n)$ [11, 13], this result immediately implies Theorem 1.5. We also verify the conjecture for another infinite family of closed hyperbolic 3–manifolds, which are the first ones that do not contain Reebless foliations given by [27].

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2. Preliminaries

In a group, we use the notation $g^a = a^{-1}ga$ for a conjugate and $[a, b] = aba^{-1}b^{-1}$ for a commutator.

We recall some results which will be useful in the proof of Theorem 1.3.

Lemma 2.1. *Let K be the Klein bottle. Then $\pi_1(K)$ contains a generalized torsion element.*

Proof. It is well known that $\pi_1(K)$ has a presentation

$$\pi_1(K) = \langle x, y \mid y^{-1}xy = x^{-1} \rangle.$$

Since $xx^y = 1$ from the relation and $x \neq 1$, x is a generalized torsion element. □

Lemma 5.1 in [12] shows:

Lemma 2.2. *If a 3–manifold M contains a projective plane, then $\pi_1(M)$ admits a torsion element, hence a generalized torsion element.*

Lemma 2.3. *If G is bi-orderable, then G has no generalized torsion element.*

Proof. Let $<$ be bi-ordering of G. Suppose that G contains a generalized torsion element g. Therefore, there exist $a_1, \ldots, a_n \in G$ such that

$$g^{a_1} g^{a_2} \ldots g^{a_n} = 1.$$

Since $g \neq 1$, we have $g > 1$ or $g < 1$. If $g > 1$, then $g^{a_i} > 1$ for any i by bi-orderability. So, the product of these conjugates is still bigger than 1, a contradiction. The case $g < 1$ is similar. □

We recall the following result due to Vinogradov [32].

Lemma 2.4. *A free product $G = G_1 * G_2 * \cdots * G_n$ of groups is bi-orderable if and only if each G_i is bi-orderable.*

Proof of Proposition 1.2. If G is bi-orderable, then G has no generalized torsion element (Lemma 2.3). Conversely, assume that G is not bi-orderable. Then it follows from Lemma 2.4 that G_1 or G_2 is not bi-orderable. Without loss of generality, we may assume G_1 is not bi-orderable. By the assumption G_1 has a generalized torsion element, which is also a generalized torsion element of G. □

3. Seifert fibered manifolds

The goal in this section is to establish Conjecture 1.1 for Seifert fibered manifolds, which may be non-orientable. Since any bi-orderable group has no generalized torsion element (Lemma 2.3), it is sufficient to show the following.

Theorem 3.1. *Let M be a Seifert fibered manifold which is possibly non-orientable. If $G = \pi_1(M)$ is not bi-orderable, then G has a generalized torsion element.*

Before proving the theorem, we recall the characterization of Seifert fibered manifolds whose fundamental groups are bi-orderable due to Boyer, Rolfsen and Wiest [5].

Theorem 3.2 ([5])**.** *Let M be a compact connected Seifert fibered manifold, and let G be its fundamental group. Then G is bi-orderable if and only if either*

1. *G is the trivial group and $M = S^3$; or*
2. *G is infinite cyclic and M is either $S^1 \times S^2$, $S^1 \tilde{\times} S^2$ or a solid Klein bottle; or*
3. *M is the total space of a locally trivial, orientable circle bundle over a surface other than S^2, P^2 or the Klein bottle.*

We should remark that in case (3) of Theorem 3.2, M is not necessarily orientable. A circle bundle over a surface is said to be *orientable* if for any loop on the base surface, its preimage under the natural projection is a torus. So, the total space of an orientable circle bundle may be non-orientable. In case (3), M is a non-orientable 3–manifold, whenever the base surface is non-orientable. For example, the trivial circle bundle over the Möbius band is a non-orientable Seifert fibered manifold, and its fundamental group is $\mathbb{Z}^2$, which is bi-orderable.

Based on the characterization in Theorem 3.2, we will show that if the fundamental group of a Seifert fibered manifold M is not bi-orderable, then it contains a generalized torsion element. The proof of Theorem 3.1 is divided into two cases according as M is orientable or not. The two cases are discussed in Subsections 3.1 and 3.2, respectively.

Let M be a compact connected Seifert fibered manifold, and G the fundamental group of M. Suppose that G is not bi-orderable hereafter.

3.1. **Proof of Theorem 3.1 for orientable Seifert fibered manifolds.** In this section, we assume that M is an orientable Seifert fibered manifold whose fundamental group G is not bi-orderable. We will look for a generalized torsion element in G.

First, we make a reduction. Since the trivial group is bi-orderable, G is non-trivial. If M is reducible, then M is either $S^1 \times S^2$ or $P^3 \# P^3$. For the first case, G is infinite cyclic, so bi-orderable. In the second case $G = \mathbb{Z}_2 * \mathbb{Z}_2$ has a torsion element. Thus in the following we assume that M is irreducible.

Fix a Seifert fibration $\mathcal{F}$ of M, and let B be a base surface obtained by identifying each fiber to a point. Then we have a natural projection $p : M \to B$. The Seifert fibration $\mathcal{F}$ gives B an orbifold structure, and we denote the base orbifold by $\mathcal{B}$.

The case where B is non-orientable is easy to settle.

Lemma 3.3. *If M is orientable and B is non-orientable, then G contains a generalized torsion element.*

Proof. Let ℓ be an orientation-reversing loop on B. Then the inverse image $p^{-1}(\ell)$ gives the Klein bottle K in M. Let T be the torus boundary of the regular neighborhood $N(K)$ of K, which is the twisted I-bundle over the Klein bottle. By Lemma 2.1, $\pi_1(N(K))$ $(= \pi_1(K))$ contains a generalized torsion element.

If the torus T is incompressible in M, then $\pi_1(N(K))$ is a subgroup of G. Hence the above generalized torsion element remains in G.

If T is compressible, then T bounds a solid torus by the irreducibility of M. Hence M is the union of the twisted I–bundle over the Klein bottle and a solid torus. Then M is either $S^1 \times S^2$, $P^3 \# P^3$, a lens space or a prism manifold. The first case is eliminated by our assumption that G is not bi-orderable. When the second case happens, $P^3 \# P^3$ is reducible, contradicting the assumption. For the remaining cases, G is finite, so it contains a torsion element. □

By Lemma 3.3, we may now assume that B is orientable. Let n be the number of exceptional fibers in $\mathcal{F}$.

Lemma 3.4. *If $n = 0$, then G contains a generalized torsion element.*

Proof. Since M is a circle bundle over B, B is S^2 by Theorem 3.2. Then M is S^3, $S^1 \times S^2$ or a lens space. Since G is not bi-orderable, M is a lens space. Hence G contains a torsion element. □

Lemma 3.5. *If G is infinite and non-abelian, and $n > 0$, then G contains a generalized torsion element.*

Proof. The canonical subgroup in the sense of [16] coincides with G. Let e be the element represented by an exceptional fiber of index α (≥ 2). By [16, II.4.7] (which needs the assumption that G is infinite), the centralizer of e is abelian, because e does not lie in the subgroup generated by a regular fiber h, which is infinite cyclic and normal.

Thus the centralizer of e is strictly smaller than G. Hence there exists an element $f \in G$ which does not commute with e. However, $e^\alpha = h$, the element represented by a regular fiber, so e^α is central in G. Thus the commutator $[e, f] \neq 1$, but $[e^\alpha, f] = 1$. We remark that $[e^\alpha, f]$ is a product of conjugates of $[e, f]$, which follows inductively from the equation

$$[e^\alpha, f] = [e^{\alpha-1}, f]^{e^{-1}}[e, f].$$

This implies that the commutator $[e, f]$ is a generalized torsion element. □

It follows from Lemma 3.4 that we can assume $n > 0$. We now separate into two cases depending upon $\partial B = \varnothing$ or not.

Case 1. $\partial B = \varnothing$.

Let g be the genus of the closed orientable surface B. If $g = 0$ and $n \leq 2$, then M is S^3, $S^1 \times S^2$ or a lens space. Since G is not bi-orderable, M is a lens space. Then, G contains a torsion element.

Suppose $g = 0$ and $n \geq 3$, or $g \geq 1$.

We claim that G is non-abelian. If G is abelian, then M is either $S^1 \times S^2$, T^3, or a lens space; see [1, p.25]. In the first two cases, G is bi-orderable. Hence M is a lens space, but this is impossible by the assumption $g = 0$ and $n \geq 3$, or $g \geq 1$.

If G is finite, then G contains a torsion element. Otherwise, the conclusion follows from Lemma 3.5.

Case 2. $\partial B \neq \varnothing$.

If B is the disk with $n = 1$, then M is a solid torus. Then G is infinite cyclic, which is bi-orderable.

If B is either the disk with $n = 2$, or an annulus with $n = 1$, then Lemma 3.5 gives the conclusion.

Except these three cases, we can choose a loop ℓ on B such that either

(1) ℓ bounds a disk with two cone points (of $\mathcal{B}$); or
(2) ℓ and one boundary component of B cobounds an annulus with one cone point (of $\mathcal{B}$),

and that the inverse image $p^{-1}(\ell)$ under the natural projection $p : M \to B$ gives a separating incompressible torus T in M.

Then the fundamental group of one side of T in M contains a generalized torsion element as above, which remains in G. This completes the proof of Theorem 3.1 for orientable Seifert fibered manifolds.

3.2. **Proof of Theorem 3.1 for non-orientable Seifert fibered manifolds.** In this section, we examine a non-orientable Seifert fibered manifold M with fundamental group G. Let n denote the number of (isolated) exceptional fibers, which are orientation-preserving in M. Exceptional fibers which are orientation-reversing, if they exist, form one-sided annuli, tori or Klein bottles in M [28, p.431]. After [25], we call such exceptional fibers *special exceptional fibers.*

Recall that we assume that G is not bi-orderable. Our goal is to find a generalized torsion element in G.

Lemma 3.6. *If $n > 0$, then M contains a generalized torsion element.*

Proof. Assume $n > 0$. Take an orientation cover $\tilde{M}$ of M. It is the unique double cover of M, which corresponds to the kernel of the surjection from G to $\mathbb{Z}_2$, sending the element of G to 0 or 1 according as the loop is orientation-preserving or not. Also, the Seifert fibration of M naturally lifts to one of $\tilde{M}$.

Let e be an isolated exceptional fiber in M. Since e is orientation-preserving, it lifts to an isolated exceptional fiber of $\tilde{M}$ with the same index.

If $\pi_1(\tilde{M})$ is not bi-orderable, then it contains a generalized torsion element by the orientable case of Theorem 3.1, which is established in Section 3.1. Since $\pi_1(\tilde{M})$ is a subgroup of G, the generalized torsion element remains in G. Therefore, we now assume that $\pi_1(\tilde{M})$ is bi-orderable, though $\pi_1(M)$ is not bi-orderable. Then, by Theorem 3.2, there are three possibilities for $\tilde{M}$ which is orientable.

Case 1. $\tilde{M}$ is S^3.

In this case, M is the quotient of S^3 under $\mathbb{Z}_2$–action. Then M would be orientable (indeed, a lens space), a contradiction; see [28, p.456].

Case 2. $\tilde{M}$ is $S^1 \times S^2$.

Since M is the quotient of $S^1 \times S^2$ under $\mathbb{Z}_2$–action, M is either $S^1 \times S^2$, $S^1 \tilde{\times} S^2$, $P^3 \# P^3$, or $S^1 \times P^2$ [28, p.457]. Since M is non-orientable, M is either $S^1 \tilde{\times} S^2$ or $S^1 \times P^2$. In the former, $\pi_1(M) = \mathbb{Z}$ is bi-orderable, contradicting the assumption. In the latter, by Lemma 2.2 $\pi_1(M)$ contains a torsion element, hence a generalized torsion element.

Case 3. $\tilde{M}$ is the total space of a locally trivial, orientable circle bundle over a surface $\tilde{B}$ other than S^2, P^2 or the Klein bottle.

Since $\tilde{M}$ is orientable, $\tilde{B}$ is also orientable. Recall that $\tilde{M}$ has an exceptional fiber in the Seifert fibration coming from M. Hence, if the fibration of $\tilde{M}$ is unique, then this is a contradiction. From the classification of Seifert fibered manifolds with non-unique fibrations [15], the only possibility of $\tilde{M}$ is $S^1 \times D^2$. Then M is a fibered solid Klein bottle [28, p.443], which contradicts the assumption that G is not bi-orderable. □

Lemma 3.7. *If M contains no exceptional fibers, then G contains a generalized torsion element.*

Proof. Since there is no exceptional fiber, M is a circle bundle over a surface B.

If B is orientable, then there exists a loop ℓ in B over which fibers cannot be coherently oriented, because M is non-orientable. Then the inverse image $p^{-1}(\ell)$ under the natural projection $p : M \to B$ gives the Klein bottle in M. If $\gamma \in G$ is represented by ℓ, then $h^{-1} = \gamma^{-1} h \gamma$, so $hh^{\gamma} = 1$, where h is represented by a regular fiber. We remark that $h \neq 1$ [5, Proposition 4.1]. Hence h is a generalized torsion element.

Assume now that B is non-orientable. If there exists a loop in B over which fibers cannot be coherently oriented, then the above argument works again. Hence

M is an orientable circle bundle over B. By Theorem 3.2, B must be either P^2 or the Klein bottle.

When $B = P^2$, there are only two orientable circle bundles over B, $S^1 \times P^2$ and $S^1 \tilde{\times} S^2$ [5, p.279]. If $M = S^1 \times P^2$, then G has a torsion element, hence a generalized torsion element (Lemma 2.2). If $M = S^1 \tilde{\times} S^2$, then G is bi-orderable, contradicting our initial assumption.

When B is the Klein bottle K, there are also two possibilities for M, $S^1 \times K$ and the non-trivial circle bundle over K. For the former, $\pi_1(K)$ is a subgroup of G. Since $\pi_1(K)$ contains a generalized torsion element by Lemma 2.1, so does G. For the latter, G has a presentation

$$G = \langle x, y, h \mid [h, x] = [h, y] = 1, x^2y^2 = h \rangle = \langle x, y \mid x^2y^2 \text{ is central} \rangle,$$

as described in [5, p.279]. Then

$$[x^2, y] = x^2yx^{-2}y^{-1} = (x^2y^2)y^{-1}x^{-2}y^{-1} = y^{-1}x^{-2}(x^2y^2)y^{-1} = 1.$$

Note $[x^2, y] = [x, y]^{x^{-1}}[x, y]$. Since there is a surjection from G onto the non-abelian group $\langle x, y \mid x^2 = y^2 = 1 \rangle = \mathbb{Z}_2 * \mathbb{Z}_2$, G is not abelian. Hence $[x, y] \neq 1$ in G. Thus $[x, y]$ is a generalized torsion element. □

It follows from Lemmas 3.6 and 3.7 that we may assume that M contains a special exceptional fiber e. Then $e^2 = h$, which is a regular fiber.

Now, the base surface B has non-empty boundary which contains reflector lines. We follow the approach of [5, Proof of Lemma 8.1 (Case 2)]. Let N be a regular neighborhood of the set of reflector lines in B, and let N_0 be a component of N. Decompose B into N_0 and $B_0 = \text{cl}(B - N_0)$. Then $N_0 \cap B_0$ is either an arc or a circle. If we put $P_0 = p^{-1}(N_0)$ and $M_0 = p^{-1}(B_0)$, then M is decomposed into P_0 and M_0 along a vertical annulus or torus, according as $N_0 \cap B_0$ is either an arc or a circle. (A vertical Klein bottle does not appear, because of the argument in the second paragraph of the proof of Lemma 3.7.) In the former case, P_0 is a fibered solid Klein bottle, and in the latter case, P_0 is the twisted I–bundle over a torus [28, pp.433-434]. In either case, $P_0 \cap M_0$ is incompressible in P_0.

If $P_0 \cap M_0$ is compressible in M_0, then P_0 is the twisted I–bundle over the torus and M_0 is a solid torus [5, p.280]. This implies that M is obtained by Dehn filling on P_0, so its fundamental group G is a quotient of $\mathbb{Z} \oplus \mathbb{Z}$. Thus G is abelian. If it is torsion-free, then it is bi-orderable, a contradiction. Hence G has a (non-trivial) torsion element, which is a generalized torsion element.

Finally, we assume that $P_0 \cap M_0$ is incompressible in M_0. Then G is the amalgamated free product of $\pi_1(P_0)$ and $\pi_1(M_0)$ over $\pi_1(P_0 \cap M_0)$. It is well known that any element in $\pi_1(P_0) - \pi_1(P_0 \cap M_0)$ does not commute with any element in $\pi_1(M_0) - \pi_1(P_0 \cap M_0)$ [20].

If the inclusion $\pi_1(P_0 \cap M_0) \to \pi_1(M_0)$ is an isomorphism, then M_0 would be the trivial I–bundle over an annulus or a torus [12, Theorems 5.2 and 10.6]. Then M is homeomorphic to P_0, so G is bi-orderable, a contradiction. Hence the inclusion $\pi_1(P_0 \cap M_0) \to \pi_1(M_0)$ is not an isomorphism.

We remark that the special exceptional fiber e lies in $\pi_1(P_0) - \pi_1(P_0 \cap M_0)$. Suppose that there exists an element $f \in \pi_1(M_0) - \pi_1(P_0 \cap M_0)$ which commutes with h. Then we have $[e, f] \neq 1$, but $[e^2, f] = [h, f] = 1$. Since $[e, f]^{e^{-1}}[e, f] = [e^2, f] = 1$, $[e, f]$ is a generalized torsion element in G. So in the following we look for such an element $f \in \pi_1(M_0) - \pi_1(P_0 \cap M_0)$.

If M_0 contains a special exceptional fiber, then it gives the desired element f. Otherwise, B_0 does not contain reflector curves. If B_0 is a disk, then M_0 is a solid torus and $P_0 \cap M_0$ is an annulus. Since the inclusion $\pi_1(P_0 \cap M_0) \to \pi_1(M_0)$ is injective, but not surjective, the core of the vertical annulus $P_0 \cap M_0$ (a regular fiber) intersects a meridian disk of M_0 more than once. This means that the core of M_0 is an exceptional fiber. Then we have a generalized torsion element by Lemma 3.6. Hence B_0 is not a disk, and we take a homotopically nontrivial loop f on B_0. As before, if the regular fibers over f cannot be oriented coherently, then there is the Klein bottle whose fundamental group contains a generalized torsion element. Otherwise, f gives the desired element commuting with h. We have thus established Theorem 3.1 for non-orientable Seifert fibered manifolds.

4. Sol manifolds

In this section we will prove:

Theorem 4.1. *Let M be a Sol manifold. If $G = \pi_1(M)$ is not bi-orderable, then G has a generalized torsion element.*

It was shown in [18, 21, 22] that if a solvable group with finite rank (i.e. there is a universal bound for the rank of finitely generated subgroups) has no generalized torsion element, then it is bi-orderable. Since a Sol manifold has a solvable fundamental group with finite rank [1, 4], the contrapositive of Theorem 4.1, hence Theorem 4.1, holds. However, we give an alternative proof by explicitly identifying a generalized torsion element in G.

The characterization of Sol manifolds with bi-orderable fundamental groups is also known by [5].

Theorem 4.2 ([5])**.** *Let M be a compact connected Sol 3–manifold with fundamental group G. Then G is bi-orderable if and only if either*

(1) *$\partial M \neq \varnothing$ and M is not the twisted I–bundle over the Klein bottle*; *or*
(2) *M is a torus bundle over the circle whose monodromy in $GL_2(\mathbb{Z})$ has at least one positive eigenvalue.*

Note that there are two twisted I–bundles over the Klein bottle; one is orientable and the other is non-orientable [10].

Proof of Theorem 4.1. Recall that M is a Sol manifold whose fundamental group G is not bi-orderable. In the following we look for a generalized torsion element in G.

Lemma 4.3. *If $\partial M \neq \varnothing$, then G contains a generalized torsion element.*

Proof. Since G is assumed to be not bi-orderable and $\partial M \neq \varnothing$, by Theorem 4.2, M is the twisted I–bundle over the Klein bottle. Then Lemma 2.1 shows that G contains a generalized torsion element. □

Thus we assume that M is closed. Following [5, p.282], there are three possibilities for M;

(1) a torus or Klein bottle bundle over the circle; or
(2) non-orientable and the union of two twisted I–bundles over the Klein bottle which are glued along their Klein bottle boundaries; or

(3) orientable and the union of two twisted I–bundles over the Klein bottle which are glued along their torus boundaries.

Except the case where M is a torus bundle over the circle, there is a π_1–injective Klein bottle in M. By Lemma 2.1, G contains a generalized torsion element. Thus we may assume that M is a torus bundle over the circle with Anosov monodromy $A \in GL_2(\mathbb{Z})$. By Theorem 4.2 and our assumption that G is not bi-orderable, A has no positive eigenvalue. (We remark that A has distinct two real eigenvalues [28, p.470].) Hence the two eigenvalues of A are negative real numbers, so $\det A = 1$ and $\mathrm{tr}(A) < -2$. Theorem 4.1 now follows from Theorem 4.4 below. □

For a torus bundle over the circle, we can find a generalized torsion element explicitly in its fundamental group under a weaker condition.

Theorem 4.4. *Let M be a torus bundle over the circle with monodromy $A \in SL_2(\mathbb{Z})$. If $\mathrm{tr}(A) < 0$, then $\pi_1(M)$ contains a generalized torsion element.*

Proof. Let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, with $ad - bc = 1$ and $a + d < 0$. Then we may assume that either $a, d \le 0$, or $a > 0$ and $d < 0$.

Now, $\pi_1(M)$ has a presentation

$$\pi_1(M) = \langle l, m, t \mid [l, m] = 1, t^{-1}lt = l^a m^b, t^{-1}mt = l^c m^d \rangle. \tag{4.1}$$

We will show that the element l is a generalized torsion element.

Since any torus fiber is π_1-injective, $l \neq 1$. From the relations, we have

$$(l^t)^{-d} = l^{-ad} m^{-bd}, (m^t)^b = l^{bc} m^{bd}. \tag{4.2}$$

From the 1st one,

$$l(l^t)^{-d} = l^{1-ad} m^{-bd}.$$

Multiplying this with the 2nd relation of (4.2), and using $ad - bc = 1$,

$$l(l^t)^{-d}(m^t)^b = 1. \tag{4.3}$$

Case 1. $a, d \le 0$

The 2nd relation of (4.1) gives

$$m^b = l^{-a} l^t.$$

From this and (4.3), we have

$$l(l^t)^{-d}(l^{-a} l^t)^t = 1.$$

Since the left hand side is a product of the conjugates of l, this shows that l is a generalized torsion element.

Case 2. $a > 0$ and $d < 0$.

(4.3) is changed to

$$l(l^{-d} m^b)^t = 1.$$

But

$$l(l^{-d} m^b)^t = l(l^{-a-d} l^a m^b)^t = l(l^{-a-d})^t (l^a m^b)^t.$$

From (4.1), $l^t = l^a m^b$. Hence

$$l(l^{-a-d})^t l^{t^2} = 1.$$

Since $a + d < 0$, the left hand side is a product of conjugates of l.

Thus we have shown that l is a generalized torsion element. □

5. Hyperbolic manifolds

Corollary 1.4 says that Conjecture 1.1 holds for any closed 3–manifold which possesses a geometric structure other than non-hyperbolic structure. In this section, we first prove Theorem 1.5, and then we verify the conjecture for some closed hyperbolic 3–manifolds introduced by Roberts, Shareshian and Stein [27].

5.1. **Cyclic branched covers of the figure-eight knot.** Let K be the figure-eight knot, and let $\Sigma_n = \Sigma_n(K)$ be the n–fold cyclic branched cover of the 3–sphere S^3 branched over K. It is known that Σ_2 is a lens space, Σ_3 is Seifert fibered, and Σ_n is hyperbolic if $n > 3$; see [11, 13]. Furthermore, any Σ_n is an L–space [26, 29], and has non-left-orderable fundamental group [9]. (A left-ordering in a group G is a strict total ordering which is invariant under left-multiplication.) In particular, $\pi_1(\Sigma_n)$ is not bi-orderable. We prove that the fundamental group of Σ_n contains a generalized torsion element when $n > 1$, from which Theorem 1.5 immediately follows.

Theorem 5.1. *The fundamental group $G = \pi_1(\Sigma_n)$ contains a generalized torsion element whenever $n > 1$.*

Proof. The Fibonacci group $F(2, m)$, introduced by Conway [8], has a presentation:

$$F(2, m) = \langle a_1, a_2, \dots, a_m \mid a_i a_{i+1} = a_{i+2} \text{ (indices modulo } m)\rangle.$$

By [11, 13], G is isomorphic to the Fibonacci group $F(2, 2n)$. Theorem 5.1 now follows from Theorem 5.2 below, which we prove a stronger statement for all Fibonacci groups. □

Recall that $F(2, m)$ is a trivial group if and only if $m = 1, 2$ [17]. When $m > 2$, we establish:

Theorem 5.2. *In the Fibonacci group $F(2, m)$ $(m > 2)$, each generator a_i is a generalized torsion element.*

Proof. It is sufficient to show that a_1 is a generalized torsion element. From the presentation, it is easy to see that $F(2, m)$ is generated by a_1 and a_2, and that there exists an automorphism, induced by a cyclic permutation on $a_1, \dots a_m$, of $F(2, m)$ which sends a_1 to any other a_i. Since $F(2, m)$ is non-trivial, we have $a_1 \neq 1$.

For simplicity, let $a = a_1$ and $b = a_2$. From the relations, $a_3 = a_1 a_2 = ab$, $a_4 = a_2 a_3 = bab$. Thus we have the expressions recursively

$$a_3 = ab, a_4 = bab,\ a_5 = ab^2ab,\ a_6 = babab^2ab, \dots.$$

We call these the *canonical expressions* of a_i's $(3 \leq i \leq m)$. In the canonical expression of a_i, neither a^{-1} nor b^{-1} appears. Let e_i denote the total exponent sum of b in the canonical expression of a_i. For example, $e_3 = 1$, $e_4 = 2$. From the relation $a_i a_{i+1} = a_{i+2}$, it is obvious that $e_i = F_{i-1}$, which is the $(i-1)$-th Fibonacci number with $F_1 = F_2 = 1$.

Hence, if we rewrite the right hand side of the equation $a_1 = a_{m-1} a_m$ into the canonical expression, then the total exponent sum of b in the expression is

$$e_{m-1} + e_m = F_{m-2} + F_{m-1} = F_m.$$

We express this equation as $a = u(a, b)$, where the word $u(a, b)$ contains only a and b, and the total exponent sum of b in $u(a, b)$ is F_m. Furthermore, take the inverse of both sides. Then we have the equation $a^{-1} = \overline{u}(a^{-1}, b^{-1})$, where the word $\overline{u}(a^{-1}, b^{-1})$ contains only a^{-1} and b^{-1}, and the total exponent sum of b in $\overline{u}(a^{-1}, b^{-1})$ is $-F_m$.

On the other hand, the relation $a_m a_1 = a_2$ enables us to express $a_m = a_2 a_1^{-1} = ba^{-1}$. Similarly, we have $a_{m-1} = a_1 a_m^{-1} = a^2 b^{-1}$ from the relations. Thus each a_i has yet another expression:

$$a_m = ba^{-1},\ a_{m-1} = a^2 b^{-1},\ a_{m-2} = ba^{-1}ba^{-2},\ a_{m-3} = a^2 b^{-1} a^2 b^{-1} ab^{-1}, \dots.$$

These are called the *non-canonical expressions* of a_i's $(3 \leq i \leq m)$.

Denote by $\overline{e}_i$ the total exponent sum of b in the non-canonical expression of a_i. For example, $\overline{e}_m = 1$, $\overline{e}_{m-1} = -1$. Then it is easy to see that $\overline{e}_i = (-1)^{m+i} F_{m+1-i}$. Moreover, in the non-canonical expression of a_i, neither a nor b^{-1} appears when $i = m, m-2, \dots$, and neither a^{-1} nor b appears when $i = m-1, m-3, \dots$. Also, if $i = m-1, m-3, \dots$, the first letter of the non-canonical expression of a_i is a, and the total exponent sum of a is at least two.

As we mentioned above, each a_i $(3 \leq i \leq m)$ has the non-canonical expression. Using the relations $a_2 = a_4 a_3^{-1}$ and $a_1 = a_3 a_2^{-1}$, we naturally extend non-canonical expressions to a_1 and a_2 so that $\overline{e}_2 = (-1)^{m+2} F_{m-1}$ and $\overline{e}_1 = (-1)^{m+1} F_m$. Then rewrite the right hand side of $a = a_1$ into the non-canonical expression to obtain $a = w_e(a, b^{-1})$ if m is even, $a = w_o(a^{-1}, b)$ if m is odd, where $w_e(a, b^{-1})$ or $w_o(a^{-1}, b)$ is the non-canonical expression of a_1 respectively. Note also that $w_e(a, b^{-1})$ contains neither a^{-1} nor b, and $w_o(a^{-1}, b)$ contains neither a nor b^{-1}.

Now we are ready to identify a generalized torsion element in $F(2, m)$.

Assume first that m is even. Then the first letter of the word $w_e(a, b^{-1})$ is a. By canceling the first letter a from both sides of the equation $a = w_e(a, b^{-1})$, we obtain a new equation $1 = w'_e(a, b^{-1})$, where $w'_e(a, b^{-1})$ still contains neither a^{-1} nor b. Moreover, $w'_e(a, b^{-1})$ contains at least one occurrence of a. Since $\overline{e}_1 = -F_m$, the total exponent sum of b in $w'_e(a, b^{-1})$ is $-F_m$. If we replace any single occurrence of a in $w'_e(a, b^{-1})$ with $a = u(a, b)$, coming from canonical expressions, then we have an equation $1 = w(a, b, b^{-1})$, where $w(a, b, b^{-1})$ contains no a^{-1}. Since the total exponent sum of b in $u(a, b)$ is F_m as mentioned before, the total exponent sum of b in $w(a, b, b^{-1})$ is $-F_m + F_m = 0$.

Let us assume that m is odd. The equation $a = w_o(a^{-1}, b)$ gives $1 = a^{-1} \cdot w_o(a^{-1}, b)$. Then replace the first a^{-1} in the right hand side with the word $\overline{u}(a^{-1}, b^{-1})$ coming from the canonical expressions. This gives $1 = \overline{u}(a^{-1}, b^{-1}) \cdot w_o(a^{-1}, b)$. The total exponent sum of b in $\overline{u}(a^{-1}, b^{-1})$ is $-F_m$, and that in $w_o(a^{-1}, b)$ is F_m. If we express the right hand side as $w(a^{-1}, b, b^{-1})$, which contains no a, then the total exponent sum of b in $w(a^{-1}, b, b^{-1})$ is $-F_m + F_m = 0$.

Claim 5.3. *The word $w(a, b, b^{-1})$ (resp. $w(a^{-1}, b, b^{-1})$) can be expressed as the product of conjugates of a (resp. a^{-1}).*

Proof. We may write

$$w(a, b, b^{-1}) = a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k},$$

where $m_1 \geq 0, m_i > 0$ $(2 \leq i \leq k)$, $n_i \neq 0$ $(i \neq k)$ and $n_1 + \cdots + n_k = 0$. Then we rewrite:

$$\begin{aligned}
w(a,b,b^{-1}) &= a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots a^{m_k} b^{n_k} \\
&= a^{m_1} (b^{n_1} a^{m_2} b^{-n_1}) b^{n_1} b^{n_2} \cdots a^{m_k} b^{n_k} \\
&= a^{m_1} (a^{m_2})^{b^{-n_1}} b^{n_1+n_2} a^{m_3} b^{n_3} \cdots a^{m_k} b^{n_k} \\
&= a^{m_1} (a^{m_2})^{b^{-n_1}} b^{n_1+n_2} a^{m_3} b^{-n_1-n_2} b^{n_1+n_2+n_3} \cdots a^{m_k} b^{n_k} \\
&= a^{m_1} (a^{m_2})^{b^{-n_1}} (a^{m_3})^{b^{-n_1-n_2}} b^{n_1+n_2+n_3} \cdots a^{m_k} b^{n_k} \\
&\vdots \\
&= a^{m_1} (a^{m_2})^{b^{-n_1}} (a^{m_3})^{b^{-n_1-n_2}} \cdots (b^{n_1+\cdots+n_{k-1}} a^{m_k} b^{n_k}) \\
&= a^{m_1} (a^{m_2})^{b^{-n_1}} (a^{m_3})^{b^{-n_1-n_2}} \cdots (b^{-n_k} a^{m_k} b^{n_k}) \\
&= a^{m_1} (a^{m_2})^{b^{-n_1}} (a^{m_3})^{b^{-n_1-n_2}} \cdots (a^{m_k})^{b^{n_k}} \\
&= a^{m_1} (a^{b^{-n_1}})^{m_2} (a^{b^{-n_1-n_2}})^{m_3} \cdots (a^{b^{n_k}})^{m_k}.
\end{aligned}$$

The proof for the word $w(a^{-1}, b, b^{-1})$ is similar. □

If a finite product of conjugates of a^{-1} becomes the identity, then, taking its inverse, we have a finite product of conjugates of a which is the identity. Thus in either case in Claim 5.3, some product of conjugates of a yields the identity. Since $a \neq 1$ in $F(2,m)$, a is a generalized torsion element. This completes the proof of Theorem 5.2. □

Remark 5.4. (1) It is known that $F(2,m)$ is a non-trivial finite group if $m = 3,4,5,7$ [17, 24]. For these cases, any non-trivial element is a torsion element, so a generalized torsion element. Furthermore, $F(2,2n+1)$ has a non-trivial torsion element [2, Proposition 3.1], but $F(2,2n)$ is torsion-free if $n > 2$.
(2) $F(2,2n)$ is the fundamental group of Σ_n. On the contrary, recently Howie and Williams [14, Theorem 2.4] proved that $F(2,2n+1)$ can be the fundamental group of a 3–manifold if and only if $n = 1, 2$ or 3.

5.2. **Other hyperbolic manifolds.** For integers p, q, m with $\gcd(p,q) = 1$, define

$$G(p,q,m) = \langle a, b, t \mid t^{-1}at = aba^{m-1}, t^{-1}bt = a^{-1}, t^p[a,b]^q = 1 \rangle. \tag{5.1}$$

In [27, Proposition 3.1], it is shown that if $m < 0$, $p > q \geq 1$, $\gcd(p,q) = 1$, then the image of any homomorphism from $G(p,q,m)$ to $\mathrm{Homeo}^+(\mathbb{R})$ is trivial. This implies that $G(p,q,m)$ is not left-orderable; see [5, Section 5]. Hence $G(p,q,m)$ is not bi-orderable.

As shown in [27], $G(p,q,m)$ is the fundamental group of a closed 3–manifold $M(p,q,m)$ which is obtained from a once-puncture torus bundle by Dehn filling. They show that if $m < -2$ and p are odd, $\gcd(p,q) = 1$ and $p \geq q \geq 1$, then $M(p,q,m)$ is hyperbolic for all except finitely many pairs (p,q) [27, Theorem A].

Under a certain condition, we can show that $G(p,q,m)$ contains a generalized torsion element.

Theorem 5.5. *If $p \geq 2q > 1$, then $G(p,q,m)$ contains a generalized torsion element.*

Proof. We will prove that the element t is a generalized torsion element.

First, $t \neq 1$, because it goes to a non-trivial element under the abelianization (we need $p > 1$ here).

The 2nd relation $a^{-1} = t^{-1}bt$ of (5.1) gives

$$[a, b] = aba^{-1}b^{-1} = t^{-1}b^{-1}tbt^{-1}btb^{-1}.$$

It is straightforward to verify that

$$\begin{aligned}[a, b]^q &= (t^{-1}b^{-1}tbt \cdot t^{-2}btb^{-1}t^2)(t^{-3}b^{-1}tbt^3 \cdot t^{-4}btb^{-1}t^4) \dots \\ &\quad (t^{-(2q-1)}b^{-1}tbt^{2q-1} \cdot t^{-2q}btb^{-1}t^{2q})t^{-2q} \\ &= (t^{bt} \cdot t^{b^{-1}t^2})(t^{bt^3} \cdot t^{b^{-1}t^4}) \dots (t^{bt^{2q-1}} \cdot t^{b^{-1}t^{2q}})t^{-2q}.\end{aligned}$$

Hence, the 3rd relation of (5.1) gives

$$t^{p-2q}(t^{bt} \cdot t^{b^{-1}t^2})(t^{bt^3} \cdot t^{b^{-1}t^4}) \dots (t^{bt^{2q-1}} \cdot t^{b^{-1}t^{2q}}) = 1.$$

If $p \geq 2q$, then the left hand side is a product of conjugates of t. Thus we have shown that the element t is a generalized torsion element. □

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