A note on L-spaces which are double branched covers of non-quasi-alternating links

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Dedicated to Taizo Kanenobu, Yasutaka Nakanishi, and Makoto Sakuma for their 60th birthdays

Abstract

Greene has given an infinite family of non-quasi-alternating links $L_{m,n}$ in which $L_{2,3}$ is homologically thin and its double branched cover $X_{2,3}$ is an L-space. In this note we show that the double branched cover $X_{m,n}$ of $L_{m,n}$ is an L-space for all $n > m \geq 2$ and $L_{m,n}$ is the unique link whose double branched cover is $X_{m,n}$.

Keywords: L-space, branched covering, Seifert fiber space, Montesinos links, extended torus links
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1. Introduction

Let M be a rational homology 3–sphere. Then its Heegaard Floer homology $\widehat{\mathrm{HF}}(M)$ introduced by Ozsváth and Szabó [28, 29] satisfies $\mathrm{rk}\widehat{\mathrm{HF}}(M) \geq |H_1(M;\mathbb{Z})|$. If the equality holds, i.e. $\widehat{\mathrm{HF}}(M) = |H_1(M;\mathbb{Z})|$, then we call M an L-space [30].

Ozsváth and Szabó [31, Lemma 3.2, Proposition 3.3] have shown that the double branched cover of any non-split alternating link, more generally quasi-alternating link, is an L-space. The set $\mathcal{Q}$ of *quasi-alternating* links is the smallest set of links containing the trivial knot, and closed under the following relation: if L admits a projection with distinguished crossing for which the two resolutions L_0 and L_1 belong to $\mathcal{Q}$, and $\det(L) = \det(L_0) + \det(L_1)$, then L belongs to $\mathcal{Q}$.

In [8] Greene gave an infinite family of non-quasi-alternating links $L_{m,n}$ ($2 \leq m < n$); see Section 4 for the definition of $L_{m,n}$. The simplest one $L_{2,3}$

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is homologically thin as computed by [1, 32], hence this gives the first example of homologically thin, non-quasi-alternating link. Moreover, $X_{2,3}$ is the first example of the L-space which is the double branched cover of (S^3 branched along) a non-quasi-alternating link ([11, Proposition 11] where $X_{2,3} = M_0$). So it is interesting to ask: is $X_{2,3}$ a double branched cover of S^3 branched along another link which is quasi-alternating? Actually the four-dimensional argument in the proof of [8, Theorem 1.3] proves the following:

Theorem 1.1 ([8]). *The L-space $X_{2,3}$ is a double branched cover of a non quasi-alternating knot $L_{2,3}$; furthermore if the double-branched cover of a link L is homeomorphic to $X_{2,3}$, then L is not quasi-alternating neither.*

We will focus on the uniqueness of such a link as $L_{2,3}$. As mentioned in [8, Section 3], $L_{m,n}$ may not be homologically thin in general. Our purpose in this note is to show:

Theorem 1.2. *Let $X_{m,n}$ be the double branched cover of $L_{m,n}$. Then $X_{m,n}$ is an L-space for any $n > m \geq 2$, and the non-quasi-alternating link $L_{m,n}$ is the unique link whose double branched cover is homeomorphic to $X_{m,n}$.*

Question 1.3. *Let M be an atoroidal L-space which is the double branched cover of a link L. Then is L the unique link whose double branched cover is homeomorphic to M?*

It should be mentioned here that Greene [9] proves that if L and L' are alternating links and their double branched covers are homeomorphic, then they are mutants. However, there exist pairs of non-mutant, non-alternating links whose double branched covers are homeomorphic. For instance, the pretzel knot $P(-2, 3, 7)$ and the torus knot $T_{3,7}$ are not mutant, but they give the same doubled branched cover $S^2(-1/2, 1/3, 1/7)$ ([2, 33]). Note that $S^2(-1/2, 1/3, 1/7)$ is not an L-space. Greene [10, Conjecture 1.5] conjectures that if a pair of links have the same double branched cover, then either both are alternating or both are non-alternating.

In [11], Greene and Watson gave further examples of homologically thin, non-quasi-alternating knots. Moreover, recently Duffield, Hoffman and Licata [5] have given an infinite family of hyperbolic L-spaces each of which has no symmetry, and none of them cannot be obtained by double branched cover of any link.

2. Tangles and Montesinos links

A *tangle* (B, t) is a pair of a 3–ball B and two disjoint arcs t properly embedded in B. We say that a tangle (B, t) is *trivial* if there is a pairwise homeomorphism from (B, t) to $(D^2 \times I, \{x_1, x_2\} \times I)$, where x_1, x_2 are distinct points. Two tangles (B, t) and (B, t') with $\partial t = \partial t'$ are *equivalent* if there is a pairwise homeomorphism $h : (B, t) \to (B, t')$ which is the identity on ∂B.

Let U be the unit 3-ball in $\mathbb{R}^3$, and take 4 points NW, NE, SE, SW on the boundary of U so that $\mathrm{NW} = (0, -\alpha, \alpha), \mathrm{NE} = (0, \alpha, \alpha), \mathrm{SE} = (0, \alpha, -\alpha), \mathrm{SW} = (0, -\alpha, -\alpha)$, where $\alpha = \frac{1}{\sqrt{2}}$. A tangle (U, t) ($\partial t = \{\mathrm{NW}, \mathrm{NE}, \mathrm{SE}, \mathrm{SW}\}$) is *rational* if it is a trivial tangle. It should be noted that a rational tangle is invariant under mutations, i.e. the π–rotations along x–, y–, and z–axis. Any rational tangle can be constructed from a sequence of integers $a_1, a_2, \ldots, a_n$ as shown in Figure 2.1, where the last horizontal twist a_n may be 0.

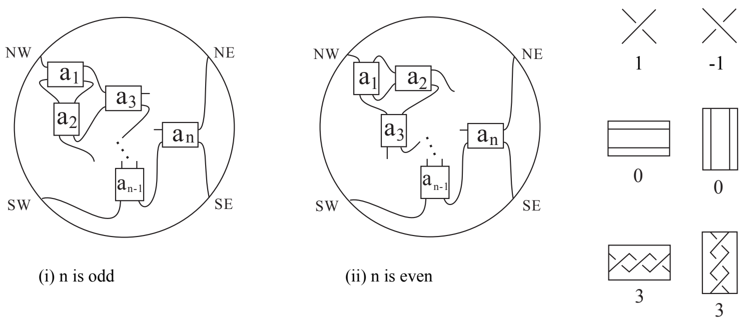
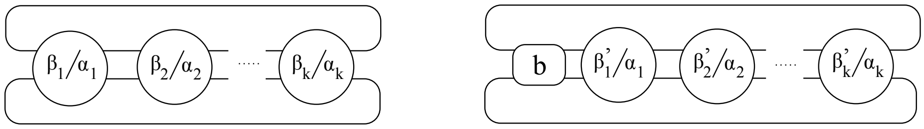


Figure 2.1: Rational tangles.

Each rational tangle can be isotoped into a position given by Figure 2.1, and parametrized by $r \in \mathbb{Q} \cup \{\infty\}$, where the rational number r is given by the continued fraction below. Thus it is convenient to denote the rational tangle corresponding to r by $R(r)$.

$$r = a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{\ddots \, a_2 + \cfrac{1}{a_1}}}$$

A *Montesinos link* $\mathcal{M}(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$ is a link which has a diagram in Figure 2.2(i), where each β_i/α_i ($\alpha_i \geq 2$) corresponds to a rational tangle $R(\beta_i/\alpha_i)$ as in Figure 2.1. Let M be the double branched cover of S^3 branched along $\mathcal{M}(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$. Then M admits a Seifert fibration over S^2 such that the preimage of the 3–ball B_i, where $R(\beta_i/\alpha_i) = (B_i, t_i)$, is a fibered solid torus whose core has a Seifert invariant β_i/α_i and index α_i. Hence $M = S^2(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$; see [21]. We can isotope the Montesinos link $\mathcal{M}(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$ to a Montesinos link with diagram in Figure 2.2(ii), where the left most box in (ii) is an integral tangle $R(b)$ and β'_i/α_i satisfies $0 < \beta'_i/\alpha_i < 1$ for $i = 1, \ldots, k$. Thus the Montesinos link $\mathcal{M}(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$ can be also expressed as $\mathcal{M}(b; \beta'_1/\alpha_1, \cdots, \beta'_k/\alpha_k)$. Correspondingly M may be expressed as $S^2(b; \beta'_1/\alpha_1, \cdots, \beta'_k/\alpha_k)$.



(i) (ii)

Figure 2.2: Montesinos link

Remark 2.1. *In general, the order of β_i/α_i is irrelevant for a Seifert fiber space $S^2(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$, but it is relevant for a Montesinos link $\mathcal{M}(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$. On the other hand, in the case where $k \le 3$ the order of β_i/α_i is irrelevant for $\mathcal{M}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$; see Claim 3.5.*

3. Seifert fiber spaces with unique double branched covering

Hodgson and Rubinstein have shown that a lens space is the double branched cover of a unique link in S^3, and this link is a two–bridge link [13, Corollary 4.12]. Furthermore, it is known that spherical Seifert fiber space is the double branched cover of a unique link in S^3, which is a Montesinos link [17, 21]. On the other hand, as we mentioned in Section 1, a Seifert fiber space M (with infinite fundamental group) may be double branched covers of two non-isotopic links L_1 and L_2; one of them is not a Montesinos link. The goal in this section is to prove the following theorem, in which, without loss of generality, we assume $2 \le \alpha_1 \le \alpha_2 \le \alpha_3$ (Remark 2.1). We denote the greatest common divisor of two integers p, q by (p, q).

Theorem 3.1. *Let M be a Seifert fiber space $S^2(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ which satisfies one of the following.*

(1) $\alpha_1 > 2$, *and* $\beta_i/\alpha_i \not\equiv \beta_j/\alpha_j \bmod 1$ *for* $i \ne j$.
(2) $\alpha_1 = 2$, $\beta_2/\alpha_2 \not\equiv \beta_3/\alpha_3 \bmod 1$, *and* $(\alpha_2, \alpha_3) > 2$.

If the double branched cover of a link $L \subset S^3$ is homeomorphic to M, then L is isotopic to a Montesinos link $\mathcal{M}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$.

Remark 3.2. *Let M be a Seifert fiber space $S^2(-1/2, 1/3, 1/7)$. Then M does not satisfy neither (1) nor (2) in Theorem 3.1, and it is the double branched cover of the pretzel knot $P(-2, 3, 7)$ and that of the torus knot $T_{3,7}$; see [2, 33].*

3.1. Double-branched covers and small Seifert fiber spaces

A link in S^3 is called an *extended torus link* if it is a union of fibers (possibly including exceptional fibers) in some Seifert fibration of S^3; see Figure 3.1(i). An extended torus link is a Seifert link, i.e. a link whose exterior is Seifert fibered, but the converse is not true. The link given by (ii) in Figure 3.1 is a Seifert link, but not an extended torus link.

Although the following result may be known to experts, for completeness, we will give a proof here.

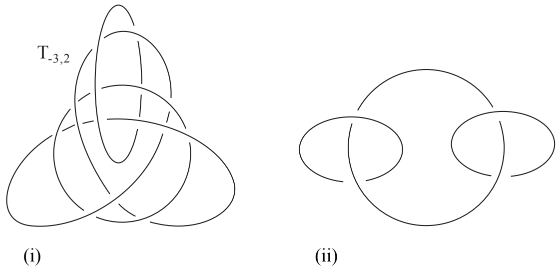


Figure 3.1: Extended torus link and Seifert link

Proposition 3.3. *Let M be a Seifert fiber space $S^2(r_1, r_2, r_3)$ with infinite fundamental group. If the double branched cover of a link $L \subset S^3$ is homeomorphic to M, then L is either a Montesinos link $\mathcal{M}(r_1, r_2, r_3)$ or an extended torus link.*

Proof. Let us put $L = k_1 \cup \cdots \cup k_n$. Let $\varphi : M \to S^3$ be the double branched covering branched along L, and $g : M \to M$ an involution satisfying $\varphi \circ g = \varphi$; $\mathrm{Fix}(g) = \widetilde{L} = \widetilde{k_1} \cup \cdots \cup \widetilde{k_n}$.

Lemma 3.4. *M has a $\langle g \rangle$–invariant Seifert fibration.*

Proof. Let H be an infinite cyclic normal subgroup of $\pi_1(M)$ generated by a regular fiber. Then we see that $\langle g \rangle$ preserves H, because Seifert fibration of M is unique up to isotopy [15, Corollary 3.12]. It follows from [19, Theorem 2.2] that we can choose a $\langle g \rangle$–invariant Seifert fibration of M. $\square$(Lemma 3.4)

By Lemma 3.4 we choose a Seifert fibration of M which is preserved by $\langle g \rangle$ so that we obtain an isomorphism $\widehat{g}$ on the base orbifold B which commutes with the Seifert fibration $\pi : M \to B$. Since M is not a prism manifold, it is sufficient to consider the case where the underlying space $|B|$ of B is S^2.

Case (1). If $\widehat{g}$ preserves the orientation of $|B| = S^2$, then Lemma 3.2 (1) in [22] shows that $\mathrm{Fix}(g)$ consists of fibers in M. Furthermore, the image of each fiber in M by the branched covering $\varphi : M \to S^3$ is a circle, thus the Seifert fibration $\mathcal{F}$ of M induces a Seifert fibration $\mathcal{F}/\langle g \rangle$ of S^3 so that φ is fiber preserving. It should be noted here that if $g(t) = t$ for a fiber t in M, since $\widehat{g}$ preserves the orientation of $|B| = S^2$, g preserves also an orientation of t, i.e. $g : t \to t$ is a rotation. Hence $L = \widetilde{L}/\langle g \rangle$ consists of fibers of the induced Seifert fibration of $S^3 = M/\langle g \rangle$, and L is an extended torus link. For details, see the argument in the proof of Lemma 5.2 in [22]. We note that L is a non-split, prime link, because its double branched cover is a small Seifert fiber space, which is irreducible.

Case (2). Assume that $\widehat{g}$ reverses the orientation of $|B| = S^2$. Since $\mathrm{Fix}(g) \neq \emptyset$, $\mathrm{Fix}(\widehat{g}) \neq \emptyset$. Thus $\widehat{g}$ is a reflection of S^2 with $\mathrm{Fix}(\widehat{g}) \cong S^1$. Suppose that $S^2 - \mathrm{Fix}(\widehat{g})$ contains cone points. Then as shown in the proof of Lemma 2.4 in [20], $M/\langle g \rangle = S^3$ would be a lens space ($\neq S^3, S^2 \times S^1$) or a nontrivial connected sum of lens spaces, a contradiction. Hence every cone point, i.e. π(exceptional fiber), lies on $\mathrm{Fix}(\widehat{g}) \cong S^1$. Apply the proof of Lemma 2.5 in [20, Section 2] to show that $L = \widetilde{L}/\langle g \rangle$ is a Montesinos link with n components. Since the double branched cover of L is a small Seifert fiber space, L has at most three branches (i.e. rational tangles). This then implies that L has at most three components, i.e. $n \leq 3$.

It remains to show that L is isotopic to $\mathcal{M}(r_1, r_2, r_3)$. Let us assume that $L = \mathcal{M}(r'_1, r'_2, r'_3)$. Then $S^2(r'_1, r'_2, r'_3)$ is orientation preservingly homeomorphic to $S^2(r_1, r_2, r_3)$. Since they have unique Seifert fibration up to isotopy [15, Corollary 3.12], Proposition 2.1 in [12] ([25, 27]) shows that we have a permutation σ of $\{1, 2, 3\}$ such that $r'_{\sigma(i)} \equiv r_i \bmod 1$ and $\Sigma_{i=1}^3 r_i = \Sigma_{i=1}^3 r'_i$.

Claim 3.5. *$\mathcal{M}(r'_1, r'_2, r'_3)$ is isotopic to $\mathcal{M}(r'_{\sigma(1)}, r'_{\sigma(2)}, r'_{\sigma(3)})$.*

PROOF. If σ is a cyclic permutation, the result follows obviously. Every transposition can be realized by the π–rotation as in Figure 3.2(ii) and mutations of rational tangles (Figure 3.2(iii)) after a suitable cyclic permutation.

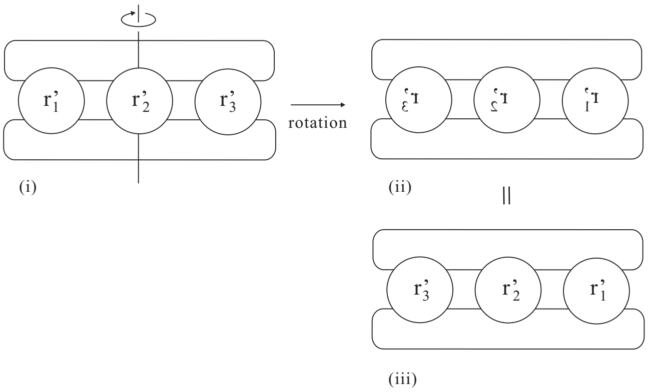


Figure 3.2: Transposition of rational tangles

□(Claim 3.5)

Claim 3.6. *$\mathcal{M}(r'_{\sigma(1)}, r'_{\sigma(2)}, r'_{\sigma(3)})$ is isotopic to $\mathcal{M}(r_1, r_2, r_3)$.*

PROOF. Let us write $r'_{\sigma(i)} = r_i + m_i$ for some integer m_i. Since $\Sigma_{i=1}^3 r'_{\sigma(i)} = \Sigma_{i=1}^3 r'_i = \Sigma_{i=1}^3 r_i$, we have $\Sigma_{i=1}^3 m_i = 0$. Apply a flype as shown in Figure 3.3, we isotope $\mathcal{M}(r'_{\sigma(1)}, r'_{\sigma(2)}, r'_{\sigma(3)})$ to $\mathcal{M}(r_1, r'_{\sigma(2)} + m_1, r'_{\sigma(3)})$. Note that if m_i is odd, we apply a mutation to get the position as in Figure 3.3.

After a sequence of flypes $\mathcal{M}(r'_{\sigma(1)}, r'_{\sigma(2)}, r'_{\sigma(3)})$ is isotoped to $\mathcal{M}(r_1, r_2, r_3 + m_1 + m_2 + m_3)$ which coincides with $\mathcal{M}(r_1, r_2, r_3)$, because $m_1 + m_2 + m_3 = 0$.
□(Claim 3.6)

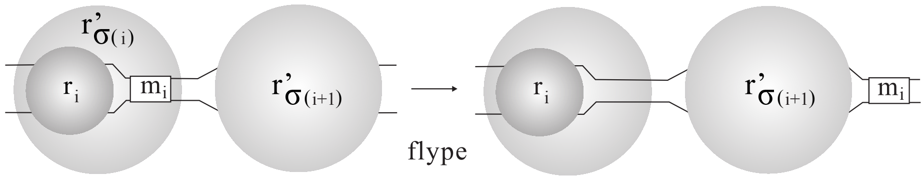


Figure 3.3: Flype

Thus $L = \mathcal{M}(r'_1, r'_2, r'_3)$ is isotopic to $\mathcal{M}(r_1, r_2, r_3)$. This completes a proof of Proposition 3.3. □(Proposition 3.3)

3.2. Proof of Theorem 3.1

Let M be a small Seifert fiber space $S^2(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ with $2 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3$. If M is spherical (i.e. $\pi_1(M)$ is finite), then $L = \mathcal{M}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ is the unique link whose double branched cover is homeomorphic to M [13, 17, 21]. So in the following we assume M is not spherical, i.e. it has infinite fundamental group. It follows from Proposition 3.3 that L is a Montesinos link $\mathcal{M}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ or an extended torus link. In the following, under the assumption of Theorem 3.1, we exclude the latter possibility.

Let $\varphi : M \to S^3$ be the branched covering projection and $g : M \to M$ the involution satisfying $\varphi \circ g = \varphi$ as in the proof of Proposition 3.3. Then following the proof of Proposition 3.3, we see that $g : M \to M$ preserves a Seifert fibration $\mathcal{F}$, and $S^3 = M/\langle g \rangle$ has a Seifert fibration $\mathcal{F}/\langle g \rangle$ so that φ sends a fiber in $\mathcal{F}$ to a fiber in $\mathcal{F}/\langle g \rangle$. The lemma below describes the relationship between indices of fibers $\widetilde{t}$ in $\mathcal{F}$ and $t = \varphi(\widetilde{t})$ in $\mathcal{F}/\langle g \rangle$. For the proof, see [22, Lemma 5.3].

Lemma 3.7. *Let $\widetilde{t}$ be a fiber of $\mathcal{F}$ which covers a fiber t of $\mathcal{F}/\langle g \rangle$, i.e. $\varphi(\widetilde{t}) = t$. Then we have:*

- *If $g(\widetilde{t}) \neq \widetilde{t}$, then* $\mathrm{index}(\widetilde{t}) = \mathrm{index}(t)$.
- *If $g(\widetilde{t}) = \widetilde{t}$ and $g|_{\widetilde{t}} : \widetilde{t} \to \widetilde{t}$ is a rotation, then* $\mathrm{index}(\widetilde{t})$ *is either* $\mathrm{index}(t)$ *or* $\mathrm{index}(t)/2$.
- *If $g(\widetilde{t}) = \widetilde{t}$ and $g|_{\widetilde{t}} : \widetilde{t} \to \widetilde{t}$ is the identity map, then* $\mathrm{index}(\widetilde{t})$ *is either* $\mathrm{index}(t)$ *or* $2\mathrm{index}(t)$.

Let $\widetilde{t_i}$ be an exceptional fiber in M whose Seifert invariant β_i/α_i $(1 \leq i \leq 3)$. Then in the first case we have the following.

Lemma 3.8. *If two exceptional fibers, say $\widetilde{t_1}$ and $\widetilde{t_2}$, cover the same exceptional fiber t in S^3, then* $\beta_1/\alpha_1 \equiv \beta_2/\alpha_2 \bmod 1$.

Proof. Let S be a section of $X = M - \bigcup_{i=1}^{3} \mathrm{int}N(\widetilde{t_i})$, where $N(\widetilde{t_i})$ is a fibered tubular neighborhood of $\widetilde{t_i}$. Let μ_i be a meridian of $N(\widetilde{t_i})$ and τ_i a regular fiber on $\partial N(\widetilde{t_i})$. Orient τ_i so that τ_1, τ_2 and τ_3 are mutually homologous in X and $\tau_i = \alpha_i \widetilde{t_i}$ ($\alpha_i \geq 2$). Choose orientation of $s_i = S \cap \partial N(\widetilde{t_i})$ so that the algebraic intersection number $\langle s_i, \tau_i \rangle$ between s_i and τ_i is $+1$. Finally choose an orientation of μ_i so that the linking number between μ_i and $\widetilde{t_i}$ is $+1$. Then $\mu_i = \alpha_i s_i + \beta_i \tau_i$. Since $g : M \to M$ and $\widehat{g} : B \to B$ preserve orientations, $g(\tau_1) = \tau_2$, $g(\mu_1) = \mu_2$. Note that $g(s_1) = s_2 + x\tau_2$ for some integer x. Then $\mu_2 = g(\mu_1) = g(\alpha_1 s_1 + \beta_1 \tau_1) = \alpha_1 g(s_1) + \beta_1 g(\tau_1) = \alpha_1(s_2 + x\tau_2) + \beta_1 \tau_2 = \alpha_1 s_2 + (\alpha_1 x + \beta_1)\tau_2$. This shows that $\beta_2/\alpha_2 = (\beta_1 + x\alpha_1)/\alpha_1 = \beta_1/\alpha_1 + x$. Hence $\beta_2/\alpha_2 \equiv \beta_1/\alpha_1 \bmod 1$. □(Lemma 3.8)

Claim 3.9. *If $\alpha_1 > 2$ and $\beta_i/\alpha_i \not\equiv \beta_j/\alpha_j \bmod 1$ for $i \neq j$, then L is not an extended torus link.*

Proof. Since $\beta_i/\alpha_i \not\equiv \beta_j/\alpha_j \bmod 1$ if $i \neq j$, Lemma 3.8 shows that there is not a pair of exceptional fibers which cover the same exceptional fiber. Since M contains three exceptional fibers and for any Seifert fibration of S^3 there are at most two exceptional fibers, there exists an exceptional fiber $\widetilde{t}$ in M which covers a regular fiber in S^3. Then the index of $\widetilde{t}$ is 2 by Lemma 3.7. This contradicts the assumption. □(Claim 3.9)

Claim 3.10. *If $\alpha_1 = 2$, $\beta_2/\alpha_2 \not\equiv \beta_3/\alpha_3 \bmod 1$, and $(\alpha_2, \alpha_3) > 2$, then L is not an extended torus link.*

Proof. Note that $\alpha_2 = \mathrm{index}(\widetilde{t_2}) \geq 3$ and $\alpha_3 = \mathrm{index}(\widetilde{t_2}) \geq 3$, for otherwise, $S^2(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ is spherical. Hence by Lemma 3.7 $\widetilde{t_2}$ and $\widetilde{t_3}$ cover exceptional fibers t_2 and t_3, respectively. Since $\beta_2/\alpha_2 \not\equiv \beta_3/\alpha_3 \bmod 1$, Lemma 3.8 shows that $t_2 \neq t_3$. Then $\mathrm{index}(t_2)$ and $\mathrm{index}(t_3)$ are relatively prime. By Lemma 3.7 α_i is one of $\mathrm{index}(t_i)$, $\mathrm{index}(t_i)/2$ or $2\mathrm{index}(t_i)$. Since $(\mathrm{index}(t_2), \mathrm{index}(t_3)) = 1$, $(\alpha_2, \alpha_3) \leq 2$. This contradicts the assumption. □(Claim 3.10)

Thus L cannot be an extended torus link, and $L = \mathcal{M}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$. □(Theorem 3.1)

Some Montesinos links are simultaneously extended torus links. In [16] Kawauchi determines pretzel knots whose double branched covers are homeomorphic to those of torus knots, and applies this to determine pretzel knots which are also torus knots. Bonahon and Siebenmann [3] completely determine Montesinos links which are also extended torus links. In its proof they analyze a Seifert fibration which is the lift of a Seifert fibration of S^3. We can also apply their arguments to show Claims 3.9 and 3.10.

4. Proof of Theorem 1.2

Let $L_{m,n}$ be a Montesinos link $\mathcal{M}(m/(m^2+1), 1/n, -m/(m^2+1))$, where $m, n \geq 2$. The double branched cover of S^3 branched along $L_{m,n}$ is a small Seifert fiber space $X_{m,n} = S^2(m/(m^2+1), 1/n, -m/(m^2+1))$. It is shown by [8, Theorem 1.3] that $L_{2,3} = \mathcal{M}(2/5, 1/3, -2/5)$ is not quasi-alternating. More generally, as mentioned in [8, Subsection 3.2], we have:

Lemma 4.1 ([8]). *The Montesinos link $L_{m,n}$ is not quasi-alternating for all $n > m \geq 2$.*

Recall that the double branched cover $X_{2,3}$ of S^3 branched along the non-quasi-alternating link $L_{2,3}$ is an L-space; see [11, Proposition 11] where $X_{2,3} = M_0$. Now let us show that $X_{m,n}$ is an L-space for all $n > m \geq 2$.

Lisca and Stipsicz [18] have shown that a Seifert fiber space M over S^2 is an L-space if and only if M does not admit a horizontal foliation. Furthermore, the combined work [6, 14, 24] classifies Seifert fiber space admitting horizontal foliations in terms of their Seifert invariants. Summarizing them we have Proposition 4.2 below, which is quoted from [4, Theorem 5.4].

For ordered triples (a_1, a_2, a_3) and (b_1, b_2, b_3), we write $(a_1, a_2, a_3) < (b_1, b_2, b_3)$ if $a_i < b_i$ for $1 \leq i \leq 3$.

Proposition 4.2. *A Seifert fiber space $S^2(b, r_1, r_2, r_3)$ $(0 < r_1 \leq r_2 \leq r_3 < 1)$ is an L-space if and only if one of the following holds.*

1. $b \geq 0$ *or* $b \leq -3$.
2. $b = -1$ *and there is no relatively prime integers* $0 < a \leq k/2$ *such that* $(r_1, r_2, r_3) < (1/k,\ a/k,\ (k-a)/k)$.
3. $b = -2$ *and there is no relatively prime integers* $0 < a \leq k/2$ *such that* $(1-r_3, 1-r_2, 1-r_1) < (1/k,\ a/k,\ (k-a)/k)$.

Using this criterion, we have:

Lemma 4.3. *$X_{m,n}$ is an L-space for all $n > m \geq 2$.*

Proof. Recall that $X_{m,n} = S^2(m/(m^2+1),\ 1/n,\ -m/(m^2+1))$, which can be expressed as $S^2(-1,\ m/(m^2+1),\ 1/n,\ 1-m/(m^2+1))$. Since $0 < m/(m^2+1),\ 1/n,\ 1-m/(m^2+1) < 1$ and $m/(m^2+1)+(1-m/(m^2+1)) = 1$, Lemma 2.3 in [23] shows that $X_{m,n}$ is an L-space for all $n > m \geq 2$. □(Lemma 4.3)

Lemma 4.4. *The non-quasi-alternating link $L_{m,n}$ is the unique link whose double branched cover is homeomorphic to $X_{m,n}$.*

Proof. Since $n > m \geq 2$, we see that $m^2+1,\ n > 2$, $m/(m^2+1) \not\equiv -m/(m^2+1) \bmod 1$, and $\pm m/(m^2+1) \not\equiv 1/n \bmod 1$. It follows from Theorem 3.1 that $L_{m,n}$ is the unique link whose double branched cover is homeomorphic to $X_{m,n}$.

□(Lemma 4.4)

Now Theorem 1.2 follows from Lemmas 4.3 and 4.4.

In [26] Núñez and Ramírez-Losada have computed Seifert invariants of branched covers of torus knots explicitly, and Gordon and Lidman [7] classify branched covers of torus knots yielding L-spaces.

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