A note on L-spaces which are double branched covers of non-quasi-alternating links

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Dedicated to Taizo Kanenobu, Yasutaka Nakanishi, and Makoto Sakuma for their 60th birthdays

Abstract
Greene has given an infinite family of non-quasi-alternating links $L_{m,n}$ in which $L_{2,3}$ is homologically thin and its double branched cover $X_{2,3}$ is an L-space. In this note we show that the double branched cover $X_{m,n}$ of $L_{m,n}$ is an L-space for all $n > m \geq 2$ and $L_{m,n}$ is the unique link whose double branched cover is $X_{m,n}$.

Keywords: L-space, branched covering, Seifert fiber space, Montesinos links, extended torus links
2000 MSC: Primary 57M12, 57M25

1. Introduction
Let $M$ be a rational homology 3–sphere. Then its Heegaard Floer homology $HF(M)$ introduced by Ozsváth and Szabó [28, 29] satisfies $\text{rk}HF(M) \geq |H_1(M;\mathbb{Z})|$. If the equality holds, i.e. $HF(M) = |H_1(M;\mathbb{Z})|$, then we call $M$ an L-space [30].

Ozsváth and Szabó [31, Lemma 3.2, Proposition 3.3] have shown that the double branched cover of any non-split alternating link, more generally quasi-alternating link, is an L-space. The set $\mathcal{Q}$ of quasi-alternating links is the smallest set of links containing the trivial knot, and closed under the following relation: if $L$ admits a projection with distinguished crossing for which the two resolutions $L_0$ and $L_1$ belong to $\mathcal{Q}$, and $\text{det}(L) = \text{det}(L_0) + \text{det}(L_1)$, then $L$ belongs to $\mathcal{Q}$.

In [8] Greene gave an infinite family of non-quasi-alternating links $L_{m,n}$ ($2 \leq m < n$); see Section 4 for the definition of $L_{m,n}$. The simplest one $L_{2,3}$
is homologically thin as computed by [1, 32], hence this gives the first example of homologically thin, non-quasi-alternating link. Moreover, $X_{2,3}$ is the first example of the L-space which is the double branched cover of ($S^3$ branched along) a non-quasi-alternating link ([11, Proposition 11] where $X_{2,3} = M_0$). So it is interesting to ask: is $X_{2,3}$ a double branched cover of $S^3$ branched along another link which is quasi-alternating? Actually the four-dimensional argument in the proof of [8, Theorem 1.3] proves the following:

**Theorem 1.1 ([8]).** The L-space $X_{2,3}$ is a double branched cover of a non quasi-alternating knot $L_{2,3}$; furthermore if the double-branched cover of a link $L$ is homeomorphic to $X_{2,3}$, then $L$ is not quasi-alternating neither.

We will focus on the uniqueness of such a link as $L_{2,3}$. As mentioned in [8, Section 3], $L_{m,n}$ may not be homologically thin in general. Our purpose in this note is to show:

**Theorem 1.2.** Let $X_{m,n}$ be the double branched cover of $L_{m,n}$. Then $X_{m,n}$ is an L-space for any $n > m \geq 2$, and the non-quasi-alternating link $L_{m,n}$ is the unique link whose double branched cover is homeomorphic to $X_{m,n}$.

**Question 1.3.** Let $M$ be an atoroidal L-space which is the double branched cover of a link $L$. Then is $L$ the unique link whose double branched cover is homeomorphic to $M$?

It should be mentioned here that Greene [9] proves that if $L$ and $L'$ are alternating links and their double branched covers are homeomorphic, then they are mutants. However, there exist pairs of non-mutant, non-alternating links whose double branched covers are homeomorphic. For instance, the pretzel knot $P(-2,3,7)$ and the torus knot $T_{3,7}$ are not mutant, but they give the same double branched cover $S^2(-1/2,1/3,1/7)$ ([2, 33]). Note that $S^2(-1/2,1/3,1/7)$ is not an L-space. Greene [10, Conjecture 1.5] conjectures that if a pair of links have the same double branched cover, then either both are alternating or both are non-alternating.

In [11], Greene and Watson gave further examples of homologically thin, non-quasi-alternating knots. Moreover, recently Duffield, Hoffman and Licata [5] have given an infinite family of hyperbolic L-spaces each of which has no symmetry, and none of them cannot be obtained by double branched cover of any link.

2. Tangles and Montesinos links

A tangle $(B, t)$ is a pair of a 3-ball $B$ and two disjoint arcs $t$ properly embedded in $B$. We say that a tangle $(B, t)$ is trivial if there is a pairwise homeomorphism from $(B, t)$ to $(D^2 \times I, \{x_1, x_2\} \times I)$, where $x_1, x_2$ are distinct points. Two tangles $(B, t)$ and $(B, t')$ with $\partial t = \partial t'$ are equivalent if there is a pairwise homeomorphism $h : (B, t) \rightarrow (B, t')$ which is the identity on $\partial B$.  


Let $U$ be the unit 3-ball in $\mathbb{R}^3$, and take 4 points NW, NE, SE, SW on the boundary of $U$ so that NW = $(0, -\alpha, \alpha)$, NE = $(0, \alpha, \alpha)$, SE = $(0, \alpha, -\alpha)$, SW = $(0, -\alpha, -\alpha)$, where $\alpha = \frac{1}{\sqrt{2}}$. A tangle $(U, t)$ ($\partial t = \{\text{NW}, \text{NE}, \text{SE}, \text{SW}\}$) is rational if it is a trivial tangle. It should be noted that a rational tangle is invariant under mutations, i.e. the $\pi$–rotations along $x$-, $y$-, and $z$–axis. Any rational tangle can be constructed from a sequence of integers $a_1, a_2, \ldots, a_n$ as shown in Figure 2.1, where the last horizontal twist $a_n$ may be 0.

![Figure 2.1: Rational tangles.](image)

Each rational tangle can be isotoped into a position given by Figure 2.1, and parametrized by $r \in \mathbb{Q} \cup \{\infty\}$, where the rational number $r$ is given by the continued fraction below. Thus it is convenient to denote the rational tangle corresponding to $r$ by $R(r)$.

$$r = a_n + \frac{1}{a_{n-1} + \frac{1}{\ddots + \frac{1}{a_2 + \frac{1}{a_1}}}}$$

A Montesinos link $\mathcal{M}(\beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$ is a link which has a diagram in Figure 2.2(i), where each $\beta_i/\alpha_i$ ($\alpha_i \geq 2$) corresponds to a rational tangle $R(\beta_i/\alpha_i)$ as in Figure 2.1. Let $M$ be the double branched cover of $S^3$ branched along $\mathcal{M}(\beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$. Then $M$ admits a Seifert fibration over $S^2$ such that the preimage of the 3–ball $B_i$, where $R(\beta_i/\alpha_i) = (B_i, t_i)$, is a fibered solid torus whose core has a Seifert invariant $\beta_i/\alpha_i$ and index $\alpha_i$. Hence $M = S^2(\beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$; see [21]. We can isotope the Montesinos link $\mathcal{M}(\beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$ to a Montesinos link with diagram in Figure 2.2(ii), where the left most box in (ii) is an integral tangle $R(b)$ and $\beta_i'/\alpha_i$ satisfies $0 < \beta_i'/\alpha_i < 1$ for $i = 1, \ldots, k$. Thus the Montesinos link $\mathcal{M}(\beta_1/\alpha_1, \ldots, \beta_k/\alpha_k)$ can be also expressed as $\mathcal{M}(b; \beta_1'/\alpha_1, \ldots, \beta_k'/\alpha_k)$. Correspondingly $M$ may be expressed as $S^2(b; \beta_1'/\alpha_1, \ldots, \beta_k'/\alpha_k)$. 

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Remark 2.1. In general, the order of $\beta_i/\alpha_i$ is irrelevant for a Seifert fiber space $S^2(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$, but it is relevant for a Montesinos link $M(\beta_1/\alpha_1, \cdots, \beta_k/\alpha_k)$. On the other hand, in the case where $k \leq 3$ the order of $\beta_i/\alpha_i$ is irrelevant for $M(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$; see Claim 3.5.

3. Seifert fiber spaces with unique double branched covering

Hodgson and Rubinstein have shown that a lens space is the double branched cover of a unique link in $S^3$, and this link is a two-bridge link [13, Corollary 4.12]. Furthermore, it is known that spherical Seifert fiber space is the double branched cover of a unique link in $S^3$, which is a Montesinos link [17, 21]. On the other hand, as we mentioned in Section 1, a Seifert fiber space $M$ (with infinite fundamental group) may be double branched covers of two non-isotopic links $L_1$ and $L_2$: one of them is not a Montesinos link. The goal in this section is to prove the following theorem, in which, without loss of generality, we assume $2 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3$ (Remark 2.1). We denote the greatest common divisor of two integers $p, q$ by $(p, q)$.

Theorem 3.1. Let $M$ be a Seifert fiber space $S^2(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$ which satisfies one of the following.

1. $\alpha_1 > 2$, and $\beta_i/\alpha_1 \neq \beta_j/\alpha_1 \mod 1$ for $i \neq j$.
2. $\alpha_1 = 2$, $\beta_2/\alpha_2 \neq \beta_3/\alpha_3 \mod 1$, and $(\alpha_2, \alpha_3) > 2$.

If the double branched cover of a link $L \subset S^3$ is homeomorphic to $M$, then $L$ is isotopic to a Montesinos link $M(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3)$.

Remark 3.2. Let $M$ be a Seifert fiber space $S^3(-1/2, 1/3, 1/7)$. Then $M$ does not satisfy neither (1) nor (2) in Theorem 3.1, and it is the double branched cover of the pretzel knot $P(-2, 3, 7)$ and that of the torus knot $T_{3,7}$; see [2, 33].

3.1. Double-branched covers and small Seifert fiber spaces

A link in $S^3$ is called an extended torus link if it is a union of fibers (possibly including exceptional fibers) in some Seifert fibration of $S^3$; see Figure 3.1(i). An extended torus link is a Seifert link, i.e. a link whose exterior is Seifert fibered, but the converse is not true. The link given by (ii) in Figure 3.1 is a Seifert link, but not an extended torus link.

Although the following result may be known to experts, for completeness, we will give a proof here.
Proposition 3.3. Let $M$ be a Seifert fiber space $S^2(r_1, r_2, r_3)$ with infinite fundamental group. If the double branched cover of a link $L \subset S^3$ is homeomorphic to $M$, then $L$ is either a Montesinos link $M(r_1, r_2, r_3)$ or an extended torus link.

Proof. Let $L = k_1 \cup \cdots \cup k_n$. Let $\phi : M \to S^3$ be the double branched covering branched along $L$, and $g : M \to M$ an involution satisfying $\phi \circ g = \phi$; $\text{Fix}(g) = \widetilde{L} = \tilde{k}_1 \cup \cdots \cup \tilde{k}_n$.

Lemma 3.4. $M$ has a $(g)$–invariant Seifert fibration.

Proof. Let $H$ be an infinite cyclic normal subgroup of $\pi_1(M)$ generated by a regular fiber. Then we see that $(g)$ preserves $H$, because Seifert fibration of $M$ is unique up to isotopy [15, Corollary 3.12]. It follows from [19, Theorem 2.2] that we can choose a $(g)$–invariant Seifert fibration of $M$.

By Lemma 3.4 we choose a Seifert fibration of $M$ which is preserved by $(g)$ so that we obtain an isomorphism $\hat{g}$ on the base orbifold $B$ which commutes with the Seifert fibration $\pi : M \to B$. Since $M$ is not a prism manifold, it is sufficient to consider the case where the underlying space $|B|$ of $B$ is $S^2$.

Case (1). If $\hat{g}$ preserves the orientation of $|B| = S^2$, then Lemma 3.2 (1) in [22] shows that $\text{Fix}(g)$ consists of fibers in $M$. Furthermore, the image of each fiber in $M$ by the branched covering $\varphi : M \to S^3$ is a circle, thus the Seifert fibration $\mathcal{F}$ of $M$ induces a Seifert fibration $\mathcal{F}/(g)$ of $S^3$ so that $\varphi$ is fiber preserving. It should be noted here that if $g(t) = t$ for a fiber $t$ in $M$, since $\hat{g}$ preserves the orientation of $|B| = S^2$, $g$ preserves also an orientation of $t$, i.e. $g : t \to t$ is a rotation. Hence $L = \hat{L}/(g)$ consists of fibers of the induced Seifert fibration of $S^3 = M/(g)$, and $L$ is an extended torus link. For details, see the argument in the proof of Lemma 5.2 in [22]. We note that $L$ is a non-split, prime link, because its double branched cover is a small Seifert fiber space, which is irreducible.

Case (2). Assume that $\hat{g}$ reverses the orientation of $|B| = S^2$. Since $\text{Fix}(g) \neq \emptyset$, $\text{Fix}(\hat{g}) \neq \emptyset$. Thus $\hat{g}$ is a reflection of $S^2$ with $\text{Fix}(\hat{g}) \cong S^1$. Suppose that $S^2 - \text{Fix}(\hat{g})$ contains cone points. Then as shown in the proof of Lemma 2.4 in [20], $M/(g) = S^3$ would be a lens space ($\neq S^3, S^2 \times S^1$) or a nontrivial connected

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3.png}
\caption{Extended torus link and Seifert link}
\end{figure}
sum of lens spaces, a contradiction. Hence every cone point, i.e. \( \pi \) (exceptional fiber), lies on \( \text{Fix}(\tilde{g}) \cong S^1 \). Apply the proof of Lemma 2.5 in [20, Section 2] to show that \( L = \tilde{L}/(g) \) is a Montesinos link with \( n \) components. Since the double branched cover of \( L \) is a small Seifert fiber space, \( L \) has at most three branches (i.e. rational tangles). This then implies that \( L \) has at most three components, i.e. \( n \leq 3 \).

It remains to show that \( L \) is isotopic to \( \mathcal{M}(r_1, r_2, r_3) \). Let us assume that \( L = \mathcal{M}(r_1', r_2', r_3') \). Then \( S^2(r_1', r_2', r_3') \) is orientation preservingly homeomorphic to \( S^2(r_1, r_2, r_3) \). Since they have unique Seifert fibration up to isotopy [15, Corollary 3.12], Proposition 2.1 in [12] ([25, 27]) shows that we have a permutation \( \sigma \) of \( \{1, 2, 3\} \) such that \( r'_\sigma(i) \equiv r_i \mod 1 \) and \( \sum_{i=1}^3 r_i = \Sigma_{i=1}^3 r'_i \).

**Claim 3.5.** \( \mathcal{M}(r_1', r_2', r_3') \) is isotopic to \( \mathcal{M}(r'_\sigma(1), r'_\sigma(2), r'_\sigma(3)) \).

**Proof.** If \( \sigma \) is a cyclic permutation, the result follows obviously. Every transposition can be realized by the \( \pi \)-rotation as in Figure 3.2(ii) and mutations of rational tangles (Figure 3.2(iii)) after a suitable cyclic permutation.

\[
\begin{align*}
\text{(i)} & \quad r_1' \quad r_2' \quad r_3' \\
\text{rotation} & \quad \Rightarrow \\
\text{(ii)} & \quad r_1' \quad r_2' \quad r_3' \\
\text{(iii)} & \quad r_1' \quad r_2' \quad r_3'
\end{align*}
\]

Figure 3.2: Transposition of rational tangles

\( \square \)(Claim 3.5)

**Claim 3.6.** \( \mathcal{M}(r'_\sigma(1), r'_\sigma(2), r'_\sigma(3)) \) is isotopic to \( \mathcal{M}(r_1, r_2, r_3) \).

**Proof.** Let us write \( r'_\sigma(i) = r_i + m_i \) for some integer \( m_i \). Since \( \Sigma_{i=1}^3 r'_\sigma(i) = \Sigma_{i=1}^3 r_i = \Sigma_{i=1}^3 m_i = 0 \), we have \( \sum_{i=1}^3 m_i = 0 \). Apply a flype as shown in Figure 3.3, we isotope \( \mathcal{M}(r'_\sigma(1), r'_\sigma(2), r'_\sigma(3)) \) to \( \mathcal{M}(r_1, r_2, r_3 + m_1, r'_\sigma(3)) \). Note that if \( m_i \) is odd, we apply a mutation to get the position as in Figure 3.3.

After a sequence of flypes \( \mathcal{M}(r'_\sigma(1), r'_\sigma(2), r'_\sigma(3)) \) is isotoped to \( \mathcal{M}(r_1, r_2, r_3 + m_1 + m_2 + m_3) \) which coincides with \( \mathcal{M}(r_1, r_2, r_3) \), because \( m_1 + m_2 + m_3 = 0 \).

\( \square \)(Claim 3.6)
Thus \( L = \mathcal{M}(r'_1, r'_2, r'_3) \) is isotopic to \( \mathcal{M}(r_1, r_2, r_3) \). This completes a proof of Proposition 3.3. \( \square \) (Proposition 3.3)

### 3.2. Proof of Theorem 3.1

Let \( M \) be a small Seifert fiber space \( S^2(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3) \) with \( 2 \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \). If \( M \) is spherical (i.e. \( \pi_1(M) \) is finite), then \( L = \mathcal{M}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3) \) is the unique link whose double branched cover is homeomorphic to \( M \) [13, 17, 21]. So in the following we assume \( M \) is not spherical, i.e. it has infinite fundamental group. It follows from Proposition 3.3 that \( L \) is a Montesinos link \( \mathcal{M}(\beta_1/\alpha_1, \beta_2/\alpha_2, \beta_3/\alpha_3) \) or an extended torus link. In the following, under the assumption of Theorem 3.1, we exclude the latter possibility.

Let \( \varphi : M \to S^3 \) be the branched covering projection and \( g : M \to M \) the involution satisfying \( \varphi \circ g = \varphi \) as in the proof of Proposition 3.3. Then following the proof of Proposition 3.3, we see that \( g : M \to M \) preserves a Seifert fibration \( \mathcal{F} \), and \( S^3 = M/\langle g \rangle \) has a Seifert fibration \( \mathcal{F}/\langle g \rangle \) so that \( \varphi \) sends a fiber in \( \mathcal{F} \) to a fiber in \( \mathcal{F}/\langle g \rangle \). The lemma below describes the relationship between indices of fibers \( \tilde{e} \) in \( \mathcal{F} \) and \( t = \varphi(\tilde{e}) \) in \( \mathcal{F}/\langle g \rangle \). For the proof, see [22, Lemma 5.3].

**Lemma 3.7.** Let \( \tilde{e} \) be a fiber of \( \mathcal{F} \) which covers a fiber \( t \) of \( \mathcal{F}/\langle g \rangle \), i.e. \( \varphi(\tilde{e}) = t \).

Then we have:

- If \( g(\tilde{e}) \neq \tilde{e} \), then \( \text{index}(\tilde{e}) = \text{index}(t) \).
- If \( g(\tilde{e}) = \tilde{e} \) and \( g|_\tilde{e} : \tilde{e} \to \tilde{e} \) is a rotation, then \( \text{index}(\tilde{e}) \) is either \( \text{index}(t) \) or \( \text{index}(t)/2 \).
- If \( g(\tilde{e}) = \tilde{e} \) and \( g|_\tilde{e} : \tilde{e} \to \tilde{e} \) is the identity map, then \( \text{index}(\tilde{e}) \) is either \( \text{index}(t) \) or \( 2\text{index}(t) \).

Let \( \tilde{e}_1 \) be an exceptional fiber in \( M \) whose Seifert invariant \( \beta_i/\alpha_i \) (\( 1 \leq i \leq 3 \)).

Then in the first case we have the following.

**Lemma 3.8.** If two exceptional fibers, say \( \tilde{e}_1 \) and \( \tilde{e}_2 \), cover the same exceptional fiber \( t \) in \( S^3 \), then \( \beta_1/\alpha_1 \equiv \beta_2/\alpha_2 \mod 1 \).
Proof. Let $S$ be a section of $X = M - \bigcup_{i=1}^{8} \text{int} \, N(t_i)$, where $N(t_i)$ is a fibered tubular neighborhood of $t_i$. Let $\mu_i$ be a meridian of $N(t_i)$ and $\tau_i$ a regular fiber on $\partial N(t_i)$. Orient $\tau_i$ so that $\tau_1, \tau_2$ and $\tau_3$ are mutually homologous in $X$ and $\tau_i = \alpha_i t_i$ ($\alpha_i \geq 2$). Choose orientation of $s_i = S \cap \partial N(t_i)$ so that the algebraic intersection number $\langle s_i, \tau_i \rangle$ between $s_i$ and $\tau_i$ is $+1$. Finally choose an orientation of $\mu_i$ so that the linking number between $\mu_i$ and $t_i$ is $+1$. Then $\mu_i = \alpha_i s_i + \beta_i \tau_i$. Since $g : M \to M$ and $\tilde{g} : B \to B$ preserve orientations, $g(\tau_1) = \tau_2$, $g(\mu_1) = \mu_2$. Note that $g(s_1) = s_2 + x \tau_2$ for some integer $x$. Then $\mu_2 = g(\mu_1) = g(\alpha_1 s_1 + \beta_1 \tau_1) = \alpha_1 g(s_1) + \beta_1 g(\tau_1) = \alpha_1 (s_2 + x \tau_2) + \beta_1 \tau_2 = \alpha_1 s_2 + (\alpha_1 x + \beta_1) \tau_2$. This shows that $\beta_2 / \alpha_2 = (\beta_1 + x \alpha_1) / \alpha_1 = \beta_1 / \alpha_1 + x$. Hence $\beta_2 / \alpha_2 \equiv \beta_1 / \alpha_1 \mod 1$. □(Lemma 3.8)

Claim 3.9. If $\alpha_1 > 2$ and $\beta_i / \alpha_i \neq \beta_j / \alpha_j \mod 1$ for $i \neq j$, then $L$ is not an extended torus link.

Proof. Since $\beta_i / \alpha_i \neq \beta_j / \alpha_j \mod 1$ if $i \neq j$, Lemma 3.8 shows that there is not a pair of exceptional fibers which cover the same exceptional fiber. Since $M$ contains three exceptional fibers and for any Seifert fibration of $S^3$ there are at most two exceptional fibers, there exists an exceptional fiber $t$ in $M$ which covers a regular fiber in $S^3$. Then the index of $t$ is 2 by Lemma 3.7. This contradicts the assumption. □(Claim 3.9)

Claim 3.10. If $\alpha_1 = 2$, $\beta_2 / \alpha_2 \neq \beta_3 / \alpha_3 \mod 1$, and $(\alpha_2, \alpha_3) > 2$, then $L$ is not an extended torus link.

Proof. Note that $\alpha_2 = \text{index}(t_2) \geq 3$ and $\alpha_3 = \text{index}(t_3) \geq 3$, for otherwise, $S^2(\beta_1 / \alpha_1, \beta_2 / \alpha_2, \beta_3 / \alpha_3)$ is spherical. Hence by Lemma 3.7 $t_2$ and $t_3$ cover exceptional fibers $t_2$ and $t_3$, respectively. Since $\beta_2 / \alpha_2 \neq \beta_3 / \alpha_3 \mod 1$, Lemma 3.8 shows that $t_2 \neq t_3$. Then $\text{index}(t_2)$ and $\text{index}(t_3)$ are relatively prime. By Lemma 3.7 $\alpha_i$ is one of $\text{index}(t_i)$, $\text{index}(t_i)/2$ or $2 \text{index}(t_i)$. Since $(\text{index}(t_2), \text{index}(t_3)) = 1, (\alpha_2, \alpha_3) \leq 2$. This contradicts the assumption. □(Claim 3.10)

Thus $L$ cannot be an extended torus link, and $L = \mathcal{M}(\beta_1 / \alpha_1, \beta_2 / \alpha_2, \beta_3 / \alpha_3)$. □(Theorem 3.1)

Some Montesinos links are simultaneously extended torus links. In [16] Kawauchi determines pretzel knots whose double branched covers are homeomorphic to those of torus knots, and applies this to determine pretzel knots which are also torus knots. Bonahon and Siebenmann [3] completely determine Montesinos links which are also extended torus links. In its proof they analyze a Seifert fibration which is the lift of a Seifert fibration of $S^3$. We can also apply their arguments to show Claims 3.9 and 3.10.
4. Proof of Theorem 1.2

Let \( L_{m,n} \) be a Montesinos link \( \mathcal{M}(m/(m^2+1), 1/n, -m/(m^2+1)) \), where \( m, n \geq 2 \). The double branched cover of \( S^3 \) branched along \( L_{m,n} \) is a small Seifert fiber space \( X_{m,n} = S^2(m/(m^2+1), 1/n, -m/(m^2+1)) \). It is shown by [8, Theorem 1.3] that \( L_{2,3} = \mathcal{M}(2/5, 1/3, -2/5) \) is not quasi-alternating. More generally, as mentioned in [8, Subsection 3.2], we have:

**Lemma 4.1 ([8]).** The Montesinos link \( L_{m,n} \) is not quasi-alternating for all \( n > m \geq 2 \).

Recall that the double branched cover \( X_{2,3} \) of \( S^3 \) branched along the non-quasi-alternating link \( L_{2,3} \) is an L-space; see [11, Proposition 11] where \( X_{2,3} = M_0 \). Now let us show that \( X_{m,n} \) is an L-space for all \( n > m \geq 2 \).

Lisca and Stipsicz [18] have shown that a Seifert fiber space \( M \) over \( S^2 \) is an L-space if and only if \( M \) does not admit a horizontal foliation. Furthermore, the combined work [6, 14, 24] classifies Seifert fiber space admitting horizontal foliations in terms of their Seifert invariants. Summarizing them we have Proposition 4.2 below, which is quoted from [4, Theorem 5.4].

For ordered triples \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\), we write \((a_1, a_2, a_3) < (b_1, b_2, b_3)\) if \( a_i < b_i \) for \( 1 \leq i \leq 3 \).

**Proposition 4.2.** A Seifert fiber space \( S^2(b_1, r_2, r_3) \) \((0 < r_1 \leq r_2 \leq r_3 < 1)\) is an L-space if and only if one of the following holds.

1. \( b > 0 \) or \( b < -3 \).
2. \( b = -1 \) and there is no relatively prime integers \( 0 < a \leq k/2 \) such that \((r_1, r_2, r_3) < (1/k, a/k, (k-a)/k)\).
3. \( b = -2 \) and there is no relatively prime integers \( 0 < a \leq k/2 \) such that \((1-r_3, 1-r_2, 1-r_1) < (1/k, a/k, (k-a)/k)\).

Using this criterion, we have:

**Lemma 4.3.** \( X_{m,n} \) is an L-space for all \( n > m \geq 2 \).

**Proof.** Recall that \( X_{m,n} = S^2(m/(m^2+1), 1/n, -m/(m^2+1)) \), which can be expressed as \( S^2(-1, m/(m^2+1), 1/n, 1-m/(m^2+1)) \). Since \( 0 < m/(m^2+1), 1/n, 1-m/(m^2+1) < 1 \) and \( m/(m^2+1) + (1-m/(m^2+1)) = 1 \), Lemma 2.3 in [23] shows that \( X_{m,n} \) is an L-space for all \( n > m \geq 2 \). \( \square \) (Lemma 4.3)

**Lemma 4.4.** The non-quasi-alternating link \( L_{m,n} \) is the unique link whose double branched cover is homeomorphic to \( X_{m,n} \).

**Proof.** Since \( n > m \geq 2 \), we see that \( m^2 + 1, n > 2, m/(m^2 + 1) \not\equiv -m/(m^2 + 1) \pmod{1}, \) and \( \pm m/(m^2 + 1) \not\equiv 1/n \pmod{1} \). It follows from Theorem 3.1 that \( L_{m,n} \) is the unique link whose double branched cover is homeomorphic to \( X_{m,n} \).
Now Theorem 1.2 follows from Lemmas 4.3 and 4.4.


Acknowledgements – I would like to thank Akio Kawauchi for pointing out that an analysis given by Bonahon and Siebenmann [3] can be used to show Claims 3.9 and 3.10, and thank Masakazu Teragaito for helpful conversations.

References


[26] V. Núñez and E. Ramírez-Losada; The trefoil knot is as universal as it can be, Topology Appl. 130 (2003), 1–17.


