THE SLOPE CONJECTURE FOR GRAPH KNOTS

KIMIHIKO MOTEGI AND TOSHIE TAKATA

Abstract. The slope conjecture proposed by Garoufalidis asserts that the Jones slopes given by the sequence of degrees of the colored Jones polynomials are boundary slopes. We verify the slope conjecture for graph knots, i.e. knots whose Gromov volume vanish.

1. Introduction

Let K be a knot in the 3–sphere S^3 and $E(K)$ the exterior $S^3 - \mathrm{int}N(K)$. Denote by (μ, λ) the preferred meridian-longitude pair of K. Then any homotopically nontrivial simple closed curves in $\partial E(K)$ represents $p[\mu] + q[\lambda] \in H_1(\partial E(K))$ for some relatively prime integers p and q. We call $p/q \in \mathbb{Q} \cup \{\infty\}$ a *boundary slope* of K if there exists an *essential* (i.e. orientable, incompressible and boundary-incompressible) surface F such that a component of ∂F represents $p[\mu] + q[\lambda] \in H_1(\partial E(K))$. Let us define:

$$bs(K) = \Big\{ r \in \mathbb{Q} \cup \{\infty\} \mid r \text{ is a boundary slope of } K \Big\}.$$

Following Hatcher [8, Corollary] $bs(K)$ is a finite subset of $\mathbb{Q} \cup \{\infty\}$ for every knot K.

The *colored Jones function* of K is a sequence of Laurent polynomials $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$ for $n \in \mathbb{N}$, where $J_{K,2}(q)$ is the ordinary Jones polynomial of K. Let $\delta_K(n)$ be the maximum degree of $J_{K,n}(q) \in \mathbb{Z}[q^{\pm 1}]$. We call $x \in \mathbb{R}$ a *cluster point* of a sequence $\{x_n\}$ if x is a limit point of a subsequence of $\{x_n\}$. We define $js(K)$ as follows:

$$js(K) = \Big\{ \text{cluster points of the sequence } \{\frac{4\delta_K(n)}{n^2}\}_{n \in \mathbb{N}} \Big\}.$$

Since the colored Jones function is q–holonomic [4, Theorem 1], Theorem 1 in [3] shows $\delta_K(n)$ is a *quadratic quasi-polynomial*, i.e.

$$\delta_K(n) = c_2(n)n^2 + c_1(n)n + c_0(n)$$

for rational valued periodic functions $c_i(n)$ with an integral period. By Lemma 1.8 in [3], $js(K)$ is the finite set of 4 times the rational values of the periodic function $c_2(n)$. Using the minimum degree $\delta^*_K(q)$ of $J_{K,n}(q)$ instead of $\delta_K(n)$, we can define:

$$js^*(K) = \Big\{ \text{cluster points of the sequence } \{\frac{4\delta^*_K(n)}{n^2}\}_{n \in \mathbb{N}} \Big\}.$$

As noted in [3, 1.4], $\delta^*_K(n) = -\delta_{K^*}(n)$ and thus $js^*(K) = -js(K^*)$, where K^* is the mirror image of K and $-X := \{-x_1, \dots, -x_m\}$ if $X = \{x_1, \dots, x_m\}$. We call an element in $js(K) \cup js^*(K)$ a *Jones slope* of K.

2010 *Mathematics Subject Classification.* Primary 57M25, 57M27

In [3], Garoufalidis proposed the following conjecture which relates Jones slopes and boundary slopes.

Conjecture 1.1 (**Slope conjecture**). *For any knot K, every Jones slope is a boundary slope, i.e.* $js(K) \cup js^*(K) \subset bs(K)$.

The conjecture was verified for torus knots, some non-alternating knots, the $(-2, 3, p)$–pretzel knots [3, Section 4], adequate knots [2, Theorem 1] and a 2–parameter family of 2–fusion knots [5, Theorem 1.1] (see also [1, Section 8]). Note that the class of adequate knots includes all alternating knots and most Montesinos knots. Recently, the conjecture was verified for iterated cables of adequate knots and for iterated torus knots [10, Theorem 1.4 and Corollary 1.5]. See also [12].

In the present note we give further supporting evidence for the sloe conjecture.

Theorem 1.2. *Suppose that K_1 and K_2 satisfy the slope conjecture, then the connected sum $K_1 \sharp K_2$ also satisfies the slope conjecture.*

A knot K is called a *graph knot* if its exterior $E(K)$ is a graph manifold, i.e. there is a family of tori which decomposes $E(K)$ into Seifert fiber spaces. This implies that any graph knot is obtained from unknots by a finite sequence of operations of cabling and connected sum; see [6, Corollary 4.2]. A graph knot can be also characterized as a knot whose Gromov volume [7, 15] vanishes [14, Corollary 1].

We will apply Theorem 1.2, [10, Proposition 3.2] (Lemma 3.1) and Lemma 3.2 to establish:

Theorem 1.3. *Every graph knot satisfies the slope conjecture.*

Acknowledgments. We would like to thank Effie Kalfagianni and Anh Tran for suggesting an error in an earlier version of the paper.

K.M. has been partially supported by JSPS Grants–in–Aid for Scientific Research (C), 26400099, The Ministry of Education, Culture, Sports, Science and Technology, Japan and Joint Research Grant of Institute of Natural Sciences at Nihon University for 2014. T.T. has been partially supported by JSPS Grants–in–Aid for Scientific Research (C), 25400094, The Ministry of Education, Culture, Sports, Science and Technology, Japan.

2. The slope conjecture and connected sum operation

In this section we prove Theorem 1.2, i.e. the positivity of the slope conjecture is preserved under connected sum.

First we describe jones slopes $js(K_1 \sharp K_2)$.

Lemma 2.1. *If $p/q \in js(K_1 \sharp K_2)$, then there exist $p_1/q_1 \in js(K_1)$ and $p_2/q_2 \in js(K_2)$ such that $p/q = p_1/q_1 + p_2/q_2$.*

Proof of Lemma 2.1. Let us put $\delta_{K_i}(n) = \alpha_i(n)n^2 + \beta_i(n)n + \gamma_i(n)$ for $i = 1, 2$, where $\alpha_i(n)$, $\beta_i(n)$ and $\gamma_i(n)$ are periodic functions with integral periods. It is known that

$$J_{K_1 \sharp K_2, n}(q) = J_{K_1,n}(q) J_{K_2,n}(q)$$

and so we have:

$$\begin{aligned}\delta_{K_1\sharp K_2}(n) &= \delta_{K_1}(n)+\delta_{K_2}(n)\\ &\quad(\alpha_1(n)+\alpha_2(n))n^2+(\beta_1(n)+\beta_2(n))n+(\gamma_1(n)+\gamma_2(n)).\end{aligned}$$

Since $\alpha_i(n)$, $\beta_i(n)$ and $\gamma_i(n)$ are periodic functions with integral periods, so are $\alpha_1(n)+\alpha_2(n)$, $\beta_1(n)+\beta_2(n)$, and $\gamma_1(n)+\gamma_2(n)$. Therefore, the Jones slope p/q of $K_1\sharp K_2$ is an element of the finite set of the rational values of $4(\alpha_1(n)+\alpha_2(n)) = 4\alpha_1(n)+4\alpha_2(n)$, and hence $p/q = p_1/q_1 + p_2/q_2$ for some Jones slopes $p_1/q_1 \in js(K_1), p_2/q_2 \in js(K_2)$. This completes a proof of Lemma 2.1. □(Lemma 2.1)

Since every composite knot has an essential meridional annulus, in the following we may assume $q_i > 0$ for $i = 1, 2$.

Lemma 2.2. *If $p_1/q_1 \in bs(K_1)$ and $p_2/q_2 \in bs(K_2)$, then $p_1/q_1 + p_2/q_2 \in bs(K_1\sharp K_2)$.*

Proof of Lemma 2.2. Let A be an essential annulus in $E(K_1\sharp K_2)$ which decomposes $E(K_1\sharp K_2)$ into $E(K_1)$ and $E(K_2)$. Let F_i be an essential surface in $E(K_i)$ and m_i the number of boundary components of F_i. (If $p_i \neq 0$, then for homological reason m_i is an even integer.) Then ∂F_i consists of m_i mutually parallel loops each of which has slope p_i/q_i ($q_i > 0$). Note that the core of A is a meridian of K_i and choose F_i so that $A \cap F_i$ consists of $m_i q_i$ spanning arcs in A. See Figure 2.1.

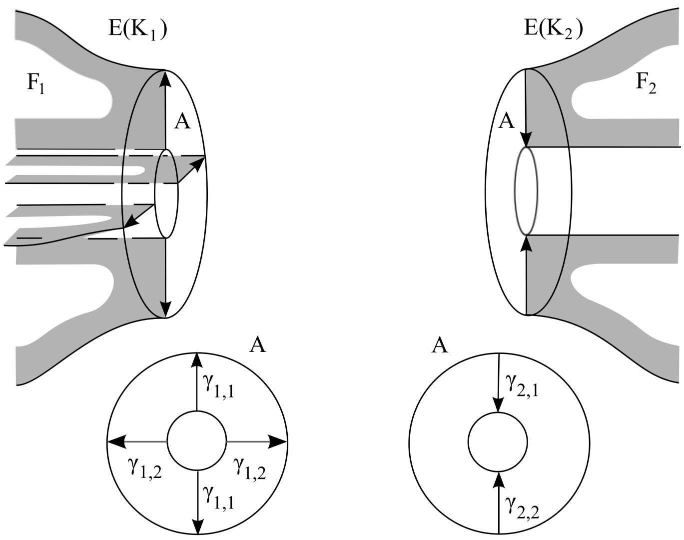

FIGURE 2.1. Essential surfaces $F_1 \subset E(K_1)$ and $F_2 \subset E(K_2)$ with $m_1 = 2,\ q_1 = 2,\ m_2 = 2,\ q_2 = 1$; $\partial F_1 = \gamma_{1,1} \cup \gamma_{1,2},\ \partial F_2 = \gamma_{2,1} \cup \gamma_{2,2}$.

Orient ∂F_i so that they run the same direction on $\partial E(K_i)$ (independent of an orientation induced from F_i) and a component of ∂F_1 and that of ∂F_2 has opposite orientations on A as in Figure 2.1. In general, $m_1q_1 \neq m_2q_2$, i.e. the number of components of $A \cap F_1$ does not coincide with that of $A \cap F_2$, so we take m_2q_2 parallel copies of F_1 and m_1q_1 parallel copies of F_2. Let us denote these (disconnected) surfaces by $m_2q_2F_1 \subset E(K_1)$ and $m_1q_1F_2 \subset E(K_2)$, respectively. We give an orientation on the boundary of $m_2q_2F_1$ (resp. $m_1q_1F_2$) so that it coincides with that of ∂F_1 (resp. ∂F_2). Since both $A \cap m_2q_2F_1$ and $A \cap m_1q_1F_2$ consist of $m_1m_2q_1q_2$ spanning arcs in A, we can connect $m_2q_2F_1$ and $m_1q_1F_2$ along the annulus A to obtain a possibly disconnected surface F' in $E(K_1 \sharp K_2)$. Note that all the components of $\partial F'$ run the same direction on $\partial E(K_1 \sharp K_2)$ with respect to the orientation given in the above.

Claim 2.3. *Each component of $F' \cap \partial E(K_1 \sharp K_2)$ has slope $p_1/q_1 + p_2/q_2$.*

Proof of Claim 2.3. Let (μ_i, λ_i) and (μ, λ) be preferred meridian-longitude pairs of K_i and $K_1 \sharp K_2$; we take $\mu_1 = \mu_2 = \mu \subset \partial A$. Orient them so that $\langle \mu_i, \lambda_i \rangle = \langle \mu, \lambda \rangle = 1$ and $\langle \mu, \partial F' \rangle = \langle \mu_1, \partial(m_2q_2F_1) \rangle = \langle \mu_2, \partial(m_1q_1F_2) \rangle > 0$, where $\langle \alpha, \beta \rangle$ denotes the algebraic intersection number between α and β. Then

$$\langle \mu, \partial F' \rangle = \langle \mu_1, \partial(m_2q_2F_1) \rangle = m_2q_2 \langle \mu_1, \partial F_1 \rangle = m_2q_2(m_1q_1) = m_1m_2q_1q_2$$

and

$$\begin{aligned} \langle \partial F', \lambda \rangle &= \langle \partial(m_2q_2F_1), \lambda_1 \rangle + \langle \partial(m_1q_1F_2), \lambda_2 \rangle \\ &= m_2q_2 \langle \partial F_1, \lambda_1 \rangle + m_1q_1 \langle \partial F_2, \lambda_2 \rangle \\ &= m_2q_2(m_1p_1) + m_1q_1(m_2p_2) \\ &= m_1m_2p_1q_2 + m_1m_2q_1p_2. \end{aligned}$$

Thus $F' \cap \partial E(K_1 \sharp K_2)$ represents

$$\begin{aligned} &(m_1m_2p_1q_2 + m_1m_2q_1p_2)[\mu] + m_1m_2q_1q_2[\lambda] \\ &= m_1m_2((p_1q_2 + q_1p_2)[\mu] + q_1q_2[\lambda]) \in H_1(\partial E(K_1 \sharp K_2)). \end{aligned}$$

Let k be the greatest common divisor of $p_1q_2 + q_1p_2$ and q_1q_2. Then $F' \cap \partial E(K_1 \sharp K_2)$ consists of m_1m_2k parallel loops each of which has slope

$$\frac{(p_1q_2 + q_1p_2)/k}{(q_1q_2)/k} = (p_1q_2 + q_1p_2)/q_1q_2 = p_1/q_1 + p_2/q_2.$$

□(Claim 2.3)

Let F be a connected component of F'. If F is non-orientable, then we take a tubular neighborhood $N(F)$ of F in $E(K_1 \sharp K_2)$ and we replace F by $\partial N(F)$, which is an orientable double cover of F and each component of $\partial N(F)$ has slope $p_1/q_1 + p_2/q_2$; for simplicity we continue to use the same symbol F to denote $\partial N(F)$. Since $F_i \subset E(K_i)$ is orientable, $F \cap E(K_i)$ consists of parallel copies of F_i for $i = 1, 2$. Note also that for each component of $F \cap E(K_i)$, its boundary component across A in the same direction.

Claim 2.4. *The surface F is essential in $E(K_1 \sharp K_2)$.*

Proof of Claim 2.4. Suppose for a contradiction that F is compressible. Let D be a compressing disk of F. If $A \cap D = \emptyset$, then D is entirely contained in $E(K_i)$ and F_i is compressible, contradicting the assumption. So in the following we assume $A \cap D \neq \emptyset$. Recall that $F \cap E(K_i)$ consists of parallel copies of F_i. Note that $A \cap F$ consists of spanning arcs in A in minimal number of components

(Figure 2.2). We may assume that D intersects A transversely and the number of components of $A \cap D$ is minimal. Then $A \cap D$ consists of circles and arcs whose endpoints belong to $A \cap F$. Since A is incompressible, we eliminate the circle components, and thus $A \cap D$ consists of arcs; see Figure 2.2.

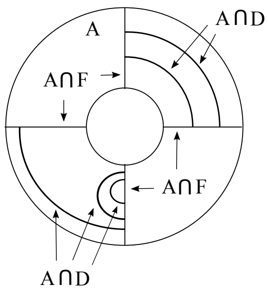

FIGURE 2.2. $A \cap F$ and $A \cap D$

Then $A \cap D$ consists of properly embedded arcs in D. Let γ be an outermost arc of $A \cap D$ in D; γ cuts off an outermost disk Δ. There are two possibilities: (i) $\partial\gamma$ is contained in a single arc τ of $A \cap F$ (Figure 2.3(i)), or (ii) γ is an arc connecting two spanning arcs τ_1 and τ_2 of $A \cap F$ (Figure 2.3(ii)).

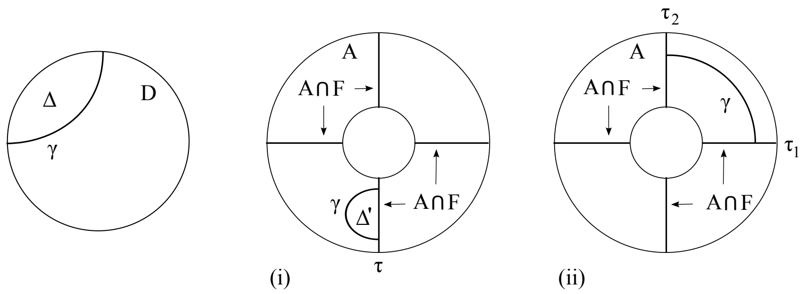

FIGURE 2.3. An outermost arc γ of $A \cap D$ in D and its possible situation in A

Suppose that $\partial\gamma$ is contained in a single arc τ of $A \cap F$ (Figure 2.3(i)). Then γ is parallel to τ; γ and τ cobound a disk $\Delta' \subset A$ (If $A \cap F$ consists of a single arc τ, then although there would be a possibility that γ starts from one side of τ and ends in the other side of τ, this cannot happen for homological reason.) Let F_Δ be a unique component of $F \cap E(K_1)$ or $F \cap E(K_2)$ intersecting $\partial\Delta$; F_Δ is a parallel copy of F_1 or F_2. Then by the incompressibility of F_i in $E(K_i)$ and the irreducibility of $E(K_i)$, the disk $\Delta \cup \Delta'$ is parallel to a disk in $\partial E(K_i)$. Thus we can isotope D so that γ is removed from $A \cap D$. This contradicts the minimality of the number of components of $A \cap D$.

Next assume that γ is an arc connecting two spanning arcs τ_1 and τ_2 of $A \cap F$ (Figure 2.3(ii)). As above we take a unique component F_Δ of $F \cap E(K_1)$ or $F \cap E(K_2)$ intersecting $\partial\Delta$; F_Δ is a parallel copy of F_1 or F_2. Since F_Δ is boundary-incompressible, τ_1 and τ_2 are contained in a single component of ∂F_Δ and run in opposite directions in A, a contradiction. It follows that F is incompressible in $E(K_1\sharp K_2)$.

Since K_i is non-trivial and F_i is not a disk for $i = 1, 2$, F is not an annulus. Hence [9, Lemma 1.10] shows that F is boundary-incompressible as well. Thus F is a desired essential surface in $E(K_1\sharp K_2)$ with boundary slope $p_1/q_1 + p_2/q_2$.

This completes a proof of Lemma 2.2. □(Lemma 2.2)

Remark 2.5. *In the above construction of the surface F', we can slide or twist several times $m_1q_1F_2$ along the annulus A before connecting with $m_2q_2F_1$ without changing its boundary slope, so F' is not unique.*

Let us turn to a proof of Theorem 1.2. Before proving the theorem, we note the following general fact.

Claim 2.6. *Let K be a knot in S^3. If $js^*(K) \subset bs(K)$, then $js(K^*) \subset bs(K^*)$.*

Proof of Claim 2.6. If $r \in js(K^*)$, then $-r \in -js(K^*) = js^*(K) \subset bs(K)$. Thus $r \in -bs(K) = bs(K^*)$. □(Claim 2.6)

Proof of Theorem 1.2. Assume first that $p/q \in js(K_1\sharp K_2)$. Then as shown in Lemma 2.1, $p/q = p_1/q_1 + p_2/q_2$ for some Jones slopes $p_1/q_1 \in js(K_1)$ and $p_2/q_2 \in js(K_2)$. Since $p_1/q_1 \in js(K_1) \subset bs(K_1)$ and $p_2/q_2 \in js(K_2) \subset bs(K_2)$ by the initial assumption, Lemma 2.2 shows that $p/q = p_1/q_1 + p_2/q_2 \in bs(K_1\sharp K_2)$.

Next assume that $p/q \in js^*(K_1\sharp K_2)$. Then $-p/q \in js((K_1\sharp K_2)^*) = js(K_1^*\sharp K_2^*)$. Since K_i satisfies the slope conjecture, $js^*(K_i) \subset bs(K_i)$, thus by Claim 2.6 $js(K_i^*) \subset bs(K_i^*)$. Apply the above argument to K_1^* and K_2^* to conclude that $-p/q \in bs(K_1^*\sharp K_2^*) = bs((K_1\sharp K_2)^*)$. Hence $p/q \in bs(K_1\sharp K_2)$. This completes a proof of Theorem 1.2. □(Theorem 1.2)

3. The slope conjecture and cabling operation

Let V be a standardly embedded solid torus in S^3 and k a 0–bridge braid in V which wraps p times in meridional direction and q times in longitudinal direction; k is a (p, q)–torus knot in S^3. In the following, we assume $q > 1$. Given a nontrivial knot K, take an orientation preserving embedding $f : V \to S^3$ such that the core of $f(V)$ is K and f sends a preferred longitude of V to that of K. Then the image $f(k)$ is called the *(p, q)–cable* of K and denoted by $C_{p,q}(K)$. Let us write $\delta_K(n) = c_2(n)n^2 + c_1(n)n + c_0(n)$, where $c_i(n)$ is a periodic function with an integral period.

An explicit cabling formula for the colored Jones functions is given in [16, Theorem 1] ([13]), and recently Kalfagianni and Tran describe how Jones slopes behave under cabling operation [10, Proposition 3.2]. Our normalization of colored Jones functions is slightly different from that in [10], and $a(n)$, $b(n)$ in [10] correspond to $c_2(n)$, $c_1(n) + \frac{1}{2}$, respectively.

Lemma 3.1 ([10])**.** *Assume that $\delta_K(n)$ has period at most 2, $c_1(n) + \frac{1}{2} \le 0$ and $4c_2(n) \neq \frac{p}{q}$ for sufficiently large n. If $r \in js(C_{p,q}(K))$, then $r = pq$ or aq^2/b for some $a/b \in js(K)$.*

The next result was essentially shown by Klaff and Shalen [11], but we give a modified proof here. See also [10, Theorem 2.2].

Lemma 3.2. *If $a/b \in bs(K)$, then $aq^2/b \in bs(C_{p,q}(K))$.*

Proof of Lemma 3.2. Since the longitudinal slope 0 is the boundary slope of a minimal genus Seifert surface, and the fiber slope pq is the boundary slope of the cabling annulus of $C_{p,q}(K)$, $0,\ pq \in bs(C_{p,q}(K))$ independent of K. So in the following we assume that $aq^2/b \neq 0,\ pq$, i.e. $a/b \neq 0,\ p/q$. In particular, K is nontrivial. Let $M_{p,q} = V - \mathrm{int}N(k)$, the standard (p,q)–cable space. We denote preferred meridian-longitude pairs of V and $N(k)$ by (μ_V, λ_V) and (μ, λ), respectively. Then $H_1(M_{p,q}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[\lambda_V]$ and $[\mu]$. Let D be a q–th punctured meridian disk of V and A an obvious annulus connecting ∂V and $\partial N(k)$. With appropriate orientations we have $[D \cap \partial V] = [\mu_V]$, $[D \cap \partial N(k)] = -q[\mu]$, $[A \cap \partial V] = p[\mu_V] + q[\lambda_V]$, $[A \cap \partial N(k)] = -pq[\mu] - [\lambda]$, and thus $[\mu_V] = q[\mu]$, $[\lambda] = q[\lambda_V]$ in $H_1(M_{p,q})$.

Let S be an oriented surface in $M_{p,q}$ representing the nontrivial homology class $(aq - bp)[D] + b[A] \in H_2(M_{p,q}, \partial M_{p,q})$. We can construct S by the "double-curve sum" of $(aq-bp)$ parallel copies of D and b parallel copies of A (i.e. cut and paste along their intersection arcs to get an embedded surface representing the desired homology class); see Figure 3.1.

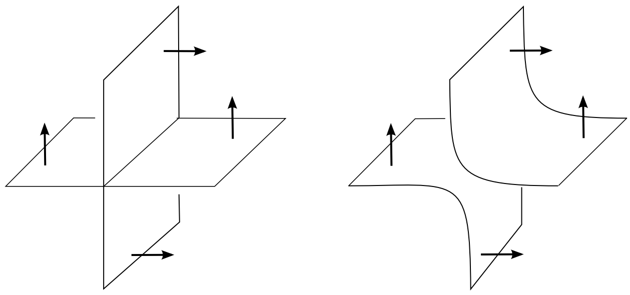

FIGURE 3.1. Double-curve sum

Then it is easy to see that each component of $S \cap \partial V$ has slope a/b, and that of $S \cap \partial N(k)$ has slope aq^2/b. If S is compressible, then after compression, we take a connected component S_0 of S which represents nontrivial homology class in $H_2(M_{p,q}, \partial M_{p,q})$. Since S_0 represents a nontrivial homology class, it is not a boundary-parallel annulus. Thus the incompressible surface S_0 is also boundary-incompressible [9, Lemma 1.10], i.e. S_0 is essential in $M_{p,q}$. Following [9, Proposition 1.11], we may assume (up to isotopy) that S_0 is horizontal (i.e. transverse to all Seifert fibers of $M_{p,q}$) or vertical (i.e. consists of Seifert fibers of $M_{p,q}$). If S_0 is vertical, then since S_0 is homologically nontrivial, $S_0 = A$. If S_0 is horizontal, then $S_0 \cap \partial V$ and $S_0 \cap \partial N(k)$ are not empty. In particular, each component of $S_0 \cap \partial V$ has slope a/b, and that of $S_0 \cap \partial N(k)$ has slope aq^2/b. Let m_0 be the number of components of ∂S_0 on ∂V.

Let $f : V \to S^3$ be an orientation preserving embedding such that $f(c) = K$ and f sends a preferred longitude of c to that of K, where c denotes a core of V. Then $C_{p,q}(K) = f(k)$ and $E(C_{p,q}(K)) = E(K) \cup f(M_{p,q})$ in which $T = \partial E(K) = \partial f(V)$ is an essential torus. Note that

$f(S_0)$ has slope a/b on $\partial E(K) = f(\partial V)$ and slope aq^2/b on $\partial E(C_{p,q}(K))$. Since $a/b \in bs(K)$, we have an essential surface $S_1 \subset E(K)$ which has m_1 boundary components each of which has slope a/b. Let us take m_1 parallel copies of $f(S_0)$ and m_0 parallel copies of S_1. Connecting them, we obtain a possibly disconnected surface F' in $E(C_{p,q}(K))$. Let F be a connected component of F'. If F is non-orientable, then as in the proof of Lemma 2.2 we replace F by $\partial N(F)$, where $N(F)$ is a tubular neighborhood of F in $E(C_{p,q}(K))$. In the latter case, we continue to use the same symbol F to denote $\partial N(F)$. Since S_0 and S_1 are orientable, $F \cap E(K)$ consists of parallel copies of S_1, and $F \cap f(M_{p,q})$ consists of parallel copies of $f(S_0)$. Applying the proof of Claim 2.4, where we use the essentiality of T instead of that of A, we see that F is incompressible in $E(C_{p,q}(K))$ with boundary slope aq^2/b. If F is not boundary-incompressible, then F is a boundary-parallel annulus [9, Lemma 1.10]. This implies that $f(S_0)(\cong S_0)$ and S_1 are planar. Since F is an annulus and S_1 is not a disk, $f(S_0)$ is an essential annulus in $f(M_{p,q})$. Thus $f(S_0)$ is a vertical annulus in the cable space $f(M_{p,q})$, and hence the boundary slope of F is the cabling slope pq, i.e. $aq^2/b = pq$, contradicting the assumption. □(Lemma 3.2)

4. The slope conjecture for graph knots

Recall that for any knot K, $\delta_K(n)$ (resp. $\delta^*_K(n)$) is a quadratic quasi-polynomial $c_2(n)n^2 + c_1(n)n + c_0(n)$ (resp. $c^*_2(n)n^2 + c^*_1(n)n + c^*_0(n)$).

Definition 4.1. We say that K satisfies *Condition* δ if

(1) $\delta_K(n)$ and $\delta^*_K(n)$ have period at most 2,
(2) $c_1(n) + \frac{1}{2} \le 0$ and $c^*_1(n) - \frac{1}{2} \ge 0$, and
(3) $4c_2(n),\ 4c^*_2(n) \in \mathbb{Z}$.

Remark 4.2. *Following the relation*

$$\delta_{K^*}(n) = -\delta^*_K(n) = -c^*_2(n)n^2 - c^*_1(n)n - c^*_0(n),$$
$$\delta^*_{K^*}(n) = -\delta_K(n) = -c_2(n)n^2 - c_1(n)n - c_0(n),$$

if K satisfies Condition δ, then its mirror image K^* also satisfies Condition δ.

Proposition 4.3. *Let $\mathcal{K}$ be the maximal set of knots each of which satisfies the slope conjecture and Condition δ. Then $\mathcal{K}$ is closed under connected sum and cabling.*

Proof of Proposition 4.3. Proposition 4.3 follows from Claims 4.4 and 4.5 below. □(Proposition 4.3)

Claim 4.4. *If $K_1, K_2 \in \mathcal{K}$, then $K_1 \sharp K_2 \in \mathcal{K}$.*

Proof of Claim 4.4. By Theorem 1.2, $K_1 \sharp K_2$ satisfies the slope conjecture. So it remains to show that $K_1 \sharp K_2$ satisfies Condition δ.

Let us write

$$\delta_{K_1}(n) = \alpha_1(n)n^2 + \beta_1(n)n + \gamma_1(n),$$
$$\delta_{K_2}(n) = \alpha_2(n)n^2 + \beta_2(n)n + \gamma_2(n).$$

Then we have:

$$\delta_{K_1\sharp K_2}(n) = (\alpha_1(n)+\alpha_2(n))n^2 + (\beta_1(n)+\beta_2(n))n + (\gamma_1(n)+\gamma_2(n)).$$

Since the common period of $\alpha_i(n), \beta_i(n)$ and $\gamma_i(n)$ is at most 2, $\alpha_1(n)+\alpha_2(n)$, $\beta_1(n)+\beta_2(n)$ and $\gamma_1(n)+\gamma_2(n)$ have period at most 2, and hence $\delta_{K_1\sharp K_2}(n)$ has also period ≤ 2. Since $\beta_1(n)+\frac{1}{2}\leq 0$ and $\beta_2(n)+\frac{1}{2}\leq 0$, $(\beta_1(n)+\beta_2(n))+\frac{1}{2}\leq 0$, which shows (2). It is obvious that $4(\alpha_1(n)+\alpha_2(n))$ is an integer. Apply the above argument to $\delta^*_{K_1}$, $\delta^*_{K_2}$ and $\delta^*_{K_1\sharp K_2}$ to see that $K_1\sharp K_2$ satisfies the remaining conditions. □(Claim 4.4)

Claim 4.5. *If $K\in\mathcal{K}$, then $C_{p,q}(K)\in\mathcal{K}$.*

Proof of Claim 4.5. Although this is shown in [10, Theorem 3.4], for convenience, we give a proof. Recall that since $K\in\mathcal{K}$, K and its mirror image K^* satisfy Condition δ (Remark 4.2).

First we show that $C_{p,q}(K)$ satisfies the slope conjecture. Assume that $r\in js(C_{p,q}(K))$. Then by Lemma 3.1 $r = pq$ or aq^2/b, where $a/b\in js(K)$. If $r = pq$, then since each boundary component of the cabling annulus has slope pq, $r\in bs(C_{p,q}(K))$. If $r = aq^2/b$, then since $a/b\in js(K)\subset bs(K)$, Lemma 3.2 shows that $r\in bs(C_{p,q}(K))$. Assume that $r\in js^*(C_{p,q}(K))$. Then $-r\in js((C_{p,q}(K))^*) = js(C_{-p,q}(K^*))$. Since $js^*(K)\subset bs(K)$ by the assumption, Claim 2.6 shows $js(K^*)\subset bs(K^*)$. Apply the above argument to K^* and its $(-p,q)$–cable to conclude that $-r\in bs(C_{-p,q}(K^*)) = bs((C_{p,q}(K))^*)$. This then implies $r\in bs(C_{p,q}(K))$. Hence $js(C_{p,q}(K))\cup js^*(C_{p,q}(K))\subset bs(C_{p,q}(K))$.

Applying Proposition 3.2 in [10] to the pair K, $C_{p,q}(K)$, and the pair K^*, $C_{-p,q}(K^*)$, we see that $C_{p,q}(K)$ also satisfies Condition δ. □(Claim 4.5)

Proof of Theorem 1.3. Let K be a graph knot. If K is the trivial knot, then it obviously satisfies the slope conjecture ($js(K)\cup js^*(K) = bs(K) = \{0\}$). Suppose that K is nontrivial. Then K is obtained from torus knots by a finite sequence of operations of cabling and connected sum; see [6, Corollary 4.2]. Garoufalidis [3, Section 4.8] proves the slope conjecture for torus knots. Actually he computes their colored Jones functions of $T_{p,q}$ $(p,q>0)$ explicitly:

$$\delta_{T_{p,q}}(n) = \frac{pq}{4}n^2 - \frac{1}{2}n - \frac{pq-2}{4} - (1+(-1)^n)\frac{(p-2)(q-2)}{8},$$
$$\delta^*_{T_{p,q}}(n) = \frac{(p-1)(q-1)}{2}n - \frac{(p-1)(q-1)}{2}.$$

Then it is easy to see that $T_{p,q}$ satisfies Condition δ. Since $\delta_{T_{-p,q}}(n) = -\delta^*_{T_{p,q}}(n)$ and $\delta^*_{T_{-p,q}}(n) = -\delta_{T_{p,q}}(n)$, any nontrivial torus knot satisfies Condition δ. It follows from Proposition 4.3 that the set of nontrivial graph knots is contained in $\mathcal{K}$. Thus any graph knot satisfies the slope conjecture. □(Theorem 1.3)

References

[1] N.M. Dunfield and S. Garoufalidis; Incompressibility criteria for spun-normal surfaces, Trans. Amer. Math. Soc., **364** (2012), 6109–6137.

[2] D. Futer, E. Kalfagianni and J. Purcell; Slopes and colored Jones polynomials of adequate knots, Proc. Amer. Math. Soc., **139** (2011), 1889–1896.

[3] S. Garoufalidis; The Jones slopes of a knot, Quantum Topology **2** (2011), 43–69.
[4] S. Garoufalidis and T.T. Le; The colored Jones function is q-holonomic, Geom. Topol. **9** (2005),1253–1293.
[5] S. Garoufalidis and R. van der Veen; Quadratic integer programming and the slope conjecture, arXiv:1405.5088.
[6] C.McA. Gordon; Dehn surgery and satellite knots, Trans. Amer. Math. Soc., **275** (1983), 687–708.
[7] M. Gromov; Volume and bounded cohomology, Inst. Hautes Études Sci. Publ. Math. **56** (1983), 213–307.
[8] A.E. Hatcher; On the boundary curves of incompressible surfaces, Pacific J. Math. **99** (1982), 373–377.
[9] A.E. Hatcher; Notes on basic 3–manifold topology (2000). Freely available at `http://www.math.cornell.edu/hatcher`
[10] E. Kalfagianni and A.T. Tran; Knot cabling and the degree of the colored Jones polynomial, New York J. Math. **21** (2015), 905–941.
[11] B. Klaff and P.B. Shalen; The diameter of the set of boundary slopes of a knot, Algebr. Geom. Topol. **6** (2006), 1095–1112.
[12] C. Lee and R. van der Veen; Slopes for pretzel knots, arXiv:1602.04546.
[13] H. Morton; The coloured Jones function and Alexander polynomial for torus knots, Math. Proc. Camb. Philos. Soc. **117** (1995), 129–135.
[14] T. Soma; The Gromov invariant of links, Invent. Math. **64** (1981), 445–454.
[15] W.P. Thurston; The geometry and topology of 3-manifolds, Lecture notes, Princeton University, 1979.
[16] R. van der Veen; A cabling formula for the colored Jones polynomial, arXiv:0807.2679.

Department of Mathematics, Nihon University, 3-25-40 Sakurajosui, Setagaya-ku, Tokyo 156–8550, Japan
E-mail address: `motegi@math.chs.nihon-u.ac.jp`

Graduate School of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819–0395, Japan
E-mail address: `ttakata@math.kyushu-u.ac.jp`